

NEWFORMS AND SPECTRAL MULTIPLICITY FOR $\Gamma_0(9)$

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ABSTRACT. The goal of this paper is to explain certain experimentally observed properties of the (cuspidal) spectrum and its associated automorphic forms (Maass waveforms) on the congruence subgroup $\Gamma_0(9)$. The first property is that the spectrum possesses multiplicities in the so-called new part, where it was previously believed to be simple. The second property is that the spectrum does not contain any “genuinely new” eigenvalues, in the sense that all eigenvalues of $\Gamma_0(9)$ appear in the spectrum of some congruence subgroup of lower level.

The main theorem in this paper gives a precise decomposition of the spectrum of $\Gamma_0(9)$ and in particular we show that the genuinely new part is empty. We also prove that there exist an infinite number of eigenvalues of $\Gamma_0(9)$ where the corresponding eigenspace is of dimension at least two and has a basis of pairs of Hecke-Maass newforms which are related to each other by a character twist. These forms are non-holomorphic analogues of modular forms with inner twists and also provide explicit (affirmative) examples of a conjecture stating that if the Hecke eigenvalues of two “generic” Maass newforms coincide on a set of primes of density $\frac{1}{2}$ then they have to be related by a character twist.

1. INTRODUCTION

1.1. A computational approach to modular forms. Since the works of Gauß and Riemann experiments have always been an important part of research in analytic number theory. However, due to both an increasing complexity of the objects studied as well as a change in the point of view of mathematical research, the experimental component did not play an as important role (relatively speaking) in the number-theoretical development in the beginning and mid 20th century (Ramanujan being a brilliant exception). Since then, beginning with the advent of the modern computer, the situation has gradually changed in favor of experiments. This is especially true in the area of L-functions and modular forms, where experiments play a prominent role today. This situation has its roots in both the discovery of effective computational methods and the development of cheap and fast processors. The latter is essentially giving the average researcher access, through a common desktop computer, to more raw computational power than that of a supercomputer some twenty years ago.

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For holomorphic modular forms the computations are greatly assisted by using associated algebraic and geometric objects, essentially enabling all necessary computations to be performed using integer arithmetic over number fields. For non-holomorphic modular forms, Maass waveforms, there is in general no algebraic or geometric theory to fall back on and it is necessary to rely on floating point calculations. The only instances of Maass waveforms where there exist explicit formulas, or even explicit information about their field of definition, are forms of CM-type coming from lifts of Hecke Größencharakteren (see e.g. [38, 16, 29]). It is believed that for a generic Maass waveform (not of CM-type) the Laplace eigenvalue as well as a certain (infinite) subset of Fourier coefficients are all algebraically independent transcendental numbers. This belief is far from proven but it is supported through extensive numerical experiments in e.g. [12]. At this point it is worth mentioning that the arithmetic properties of *Harmonic weak* Maass forms (which we do not treat here) are completely different. See e.g. [13, 14, 15].

In view of this (apparent) non-arithmeticity it is not surprising that the experimental and computational component of the fundamental research is even stronger in the area of non-holomorphic than holomorphic modular (and automorphic) forms. The very first computations of (non CM-type) Maass waveforms were performed in the 70's. For an overview of the early history and a collection of tables see [26, Appendix C]. In the 80's and 90's more efficient algorithms were developed by Stark and Hejhal (both separately and in collaboration). See e.g. [57] and [28]. The algorithm developed by Hejhal in [28], based on so-called implicit automorphy, has been generalized in various directions by the author [60, 61], Avelin [4, 6, 5] and Then [63, 64].

1.2. Experimental observations of the spectrum. One of the features observed in the early computations of Maass cusp forms for the modular group, $SL_2(\mathbb{Z})$, was that the spectrum seemed to be simple (cf. e.g. [17, 10, 27, 58]). This has later been verified (numerically) for a large number of eigenvalues for the modular group (cf. e.g. [63]), as well as for certain parts of the spectrum corresponding to congruence subgroups, the so called “new” part (in the sense of Atkin-Lehner [3]). Based on these computations and “general principles” (e.g. the Wigner-von Neumann Theorem [66]) it has since been widely believed that the new part of the spectrum on congruence subgroups is simple.

During numerical investigations in connection with [60, 61] two cases contradicting this belief were discovered. The first such case was for groups $\Gamma_0(p)$, with p prime, and Nebentypus given by the quadratic residue symbol modulo p . In this case it is easy to see that φ and $\overline{\varphi}$ (where $\overline{\varphi}$ is the complex conjugate of φ) have the same transformation properties, but are not necessarily identical. In fact, in this case $\varphi = \overline{\varphi}$ holds only for CM-type forms and these correspond to a subset of the spectrum of density zero. See e.g. [60, Rem. 1.2.3]. An analogous but more intricate argument provides similar examples of

multiplicity for composite levels and certain types of non-trivial characters, cf. e.g. Booker and Strömbergsson [11, Sec. 3.4].

The purpose of this paper is to explain another kind of multiplicity occurring for $\Gamma_0(9)$ with trivial Nebentypus. The Laplace spectrum for this group was studied (numerically) independently by the author (in connection with [60]) and by D. Farmer and S. Lemurell (in connection with [19]).

TABLE 1. Eigenvalues λ of $\Gamma_0(9)$ with $\lambda = \frac{1}{4} + R^2$ and $R < 10$

R (odd)	new on	R (even)	new on
3.5360020929376	Γ^3	3.5360020929376	Γ^3
4.3880535632221	$\Gamma_0(3)$		
		5.0987419087295	$\Gamma_0(3)$
5.5040566796766	Γ^3	5.5040566796765	Γ^3
6.1205755330872	$\Gamma_0(3)$		
6.6465813556094	Γ^3	6.6465813556097	Γ^3
6.7574152777543	$\Gamma_0(3)$		
7.4317991717218	Γ^3	7.4317991717219	Γ^3
		8.038861203863	$\Gamma_0(3)$
7.7581331950210	$\Gamma_0(3)$		
8.193035931685	$\Gamma_0(3)$		
8.698342956468	Γ^3	8.6983429564685	Γ^3
		8.7782823935563	$\Gamma_0(3)$
9.0800693497016	Γ^3	9.080069349694	Γ^3
9.2923793282248	$\Gamma_0(3)$		
9.5336952613536	$\Gamma_0(1)$		
		9.74374939916	$\Gamma_0(3)$

The first few computed spectral parameters for $\Gamma_0(9)$, together with some additional information, are listed in Table 1. For the purpose of numerical computations it is advantageous to desymmetrize the spectrum and split it into its “odd” and “even” parts, where an eigenvalue is said to be odd or even if the corresponding eigenfunction is odd or even, respectively, with respect to the reflection in the imaginary axis, $z \mapsto -\bar{z}$. From the numerical data alone it is clear that something interesting is happening here, since there are eigenvalues showing up as both odd and even. Although eigenvalues associated to oldforms always appear with multiplicity at least two, they can never appear in both the odd and the even part of the spectrum simultaneously. A closer inspection of the spectrum of the modular group and of $\Gamma_0(3)$ showed that these (at that point unexplained) even and odd multiplicities were all associated to newforms, making the situation even more intriguing. At this

point it should also be mentioned that the pattern shown in Table 1 seems to continue, at least up to $R = 50$. Based on the numerical data alone, we were tempted to *conjecture* that the entire part of the spectrum not associated with $\mathrm{PSL}_2(\mathbb{Z})$ or $\Gamma_0(3)$ (that is oldforms and twists) comes with multiplicity greater than or equal to two. The real key to proving this conjecture was the discovery that all the new eigenvalues also appeared in the spectrum of another congruence group: Γ^3 (and also there with multiplicities). It is therefore equally tempting to conjecture that all of the spectrum of $\Gamma_0(9)$ is accounted for by looking at the modular group, $\Gamma_0(3)$ and Γ^3 .

1.3. Formulation and discussion of the main theorem. In the course of proving the conjectures mentioned above (cf. Corollary 1.2 and Proposition 1.3), we arrived at Theorem 1.1 which is presented below and is given in more detail in Section 4. This theorem gives a complete description of the space of Maass waveforms on $\Gamma_0(9)$ in terms of forms on congruence subgroups of lower level.

Theorem 1.1. *The space of Maass waveforms on $\Gamma_0(9)$ can be decomposed as a direct sum of spaces of forms of the following four types: oldforms, twists of forms on the modular group, twists of forms on $\Gamma_0(3)$ and forms related to newforms on the group Γ^3 . Furthermore, the last constituent is non-empty for an infinite number of eigenvalues.*

The precise meaning of “oldforms” and “twists” in this situation will be explained in Sections 2.4 and 2.5. We say that a Maass waveform on a congruence subgroup, which is neither a twist of, nor associated to, a Maass waveform on a group of lower level is *genuinely new*. This notion will be discussed further in Section 4 and we can formulate the following consequence of Theorem 1.1.

Corollary 1.2. *There are no genuinely new Maass waveforms on $\Gamma_0(9)$.*

By exploiting properties of the group Γ^3 , in particular that it has two non-commuting involutions $z \mapsto z + 1$ and $z \mapsto -\bar{z}$, we will also show the following proposition in Section 3.3.

Proposition 1.3. *Two fifth of the new part of the spectrum of $\Gamma_0(9)$ has multiplicity at least two.*

To prove the main theorem we calculate all different types of maps taking forms on $\mathrm{PSL}_2(\mathbb{Z})$, $\Gamma_0(3)$ and Γ^3 to forms on $\Gamma_0(9)$. We then show that the corresponding images inside the space of Maass waveforms on $\Gamma_0(9)$, are all disjoint (and in fact orthogonal). To show that the complement of the sum of these images is empty we write down the Selberg trace formula explicitly for all the groups involved. Using an inclusion–exclusion argument and subtracting off all contributions from forms associated to lower level subgroups we are then able to show that the remaining part vanishes completely.

Note that, in order to prove a weaker version of Proposition 1.3 with “A positive proportion” replaced by “An infinite number” we would not need to use the full Selberg trace formula; using Weyl’s law (which essentially amounts to the main term) would be enough.

In Section 7.1 we give a brief discussion on a representation-theoretical interpretation of the main theorem. We also sketch an alternative proof of Corollary 1.2 using elementary representation theory of finite groups, together with the theory of automorphic representations of adèle groups.

1.4. Applications. It turns out that the construction we use to prove the existence of forms with multiple Laplace eigenvalues is compatible with the Hecke theory for Maass waveforms on $\Gamma_0(9)$ (see Sections 2.4 and 6). In Section 6 we will prove the following Proposition.

Proposition 1.4. *There exist an infinite number of pairs $\{F^+, F^-\}$ of Maass newforms on $\Gamma_0(9)$ with the property that the Hecke eigenvalues $c^+(n)$ and $c^-(n)$ of F^+ and F^- are related through $c^-(n) = \left(\frac{n}{3}\right) c^+(n)$ for all n relatively prime to 3.*

A pair of functions as in Proposition 1.4 have Hecke eigenvalues which coincide for a set of primes of density $\frac{1}{2}$, but unless $c^+(n) = c^-(n) = 0$ for all $n \equiv -1 \pmod{3}$ they are not identical (this is essentially the condition of being a CM-type form). It therefore provides an example of a conjecture of Rajan [44, 45, 46] (here formulated in the setting of Maass waveforms): if two generic (non CM-type) cuspidal Maass newforms have Hecke eigenvalues agreeing for a set of primes of positive density then one of them must be a twist of the other by a Dirichlet character. The following variant of this conjecture was proven by Ramakrishnan [49, Cor. 4.1.3]: if two cuspidal Maass newforms f and g have Hecke eigenvalues whose squares are equal then they are in fact related through a twist by a Dirichlet character.

The following multiplicity one theorem was proven by Ramakrishnan [48]:

Theorem 1.5 (Ramakrishnan). *Let f and g be two Maass cusp forms with Hecke eigenvalues $a_f(p)$ and $a_g(p)$. If the proportion of primes p for which $a_f(p) \neq a_g(p)$ is less than $\frac{1}{8}$ then $f = g$.*

By examples constructed by Serre and others (see e.g. [47]) it is known that in the general formulation above, the number $\frac{1}{8}$ is sharp. However, the constructed examples are all of special types (corresponding to CM-type forms), and by the analogous results in the l -adic setting by Rajan [44] it is in fact expected (cf. e.g. [46, p. 189]) that the number $\frac{1}{8}$ can be replaced by $\frac{1}{2}$ under the assumption that f and g are of general (non-CM) type. The existence of the forms in Proposition 1.4 proves the following corollary.

Corollary 1.6. *It is not possible to replace $\frac{1}{8}$ in the above theorem with any number greater than $\frac{1}{2}$ even under the assumption that f and g are of general type.*

By using the same methods as presented in this paper it is possible to prove the analogue of Theorem 1.1 for $\mathcal{M}_k(9)$, the space of holomorphic modular forms of weight k on $\Gamma_0(9)$ (and similarly for the space of cusp forms $\mathcal{S}_k(9)$). The only modification to the proof is that we do not need to use the Selberg trace formula but can instead use a standard dimension formula for spaces of holomorphic modular forms, cf. e.g. [39, Thm. 2.5.2].

Theorem 1.7. *The decomposition given in Theorem 1.1 holds also for holomorphic modular forms.*

For the benefit of the reader we also formulate the following immediate consequence of the above theorem.

Corollary 1.8. *There are no genuinely new holomorphic modular forms on $\Gamma_0(9)$.*

We should also mention that there is one part of this paper which does not go through directly to the setting of holomorphic modular forms. Since the reflection $z \mapsto -\bar{z}$ is anti-holomorphic we can not use this operator to force multiplicity. Without this tool we are not able to prove that the operator $T^{1/3}$ acting on the space of newforms in $\mathcal{M}_k(9)$ possesses non-zero eigenfunctions with eigenvalue ζ_3 as well as ζ_3^2 . We can therefore only say that there exists an infinite sequence of holomorphic newforms on $\Gamma_0(9)$ which are either invariant under twisting by $\chi = \left(\frac{\cdot}{3}\right)$ or appear in pairs f, g with $f_\chi = g$. The first case corresponds to forms of CM type and that the second case corresponds to forms with inner twists. For a precise definition and results related to inner twists see e.g. [51, 52, 40, 35]. For the sake of clarity we give a simple example of the situation for holomorphic modular forms.

Example 1.9. Consider the space of cusp forms of weight 12 on $\Gamma_0(9)$. Using e.g. Sage [59] we see that $\mathcal{S}_{12}^n(9) = \{f, g, h\}$ with the following q -expansions ($q = e^{2\pi i\tau}$ for $\tau \in \mathbb{H}$):

$$\begin{aligned} f(\tau) &= q + 24q^2 - 1472q^4 - 4830q^5 - 16744q^7 - 84480q^8 - 115920q^{10} - \dots, \\ g(\tau) &= q - 78q^2 + 4036q^4 + 5370q^5 - 27760q^7 - 155064q^8 - 418860q^{10} - \dots, \\ h(\tau) &= q + xq^2 + 472q^4 + 224xq^5 + 58100q^7 - 1576xq^8 + 564480q^{10} - \dots \end{aligned}$$

where $x^2 - 2520 = 0$. The newform h has two embeddings, h^\pm , given by $\sigma^\pm : a + b\sqrt{2520} \mapsto a \pm b\sqrt{2520}$. The embeddings act on the Fourier coefficients in the usual way, i.e. if $h = \sum a_n q^n$ then $h_{\sigma^\pm} = \sum \sigma^\pm(a_n) q^n$. These expansions can be extended as far as necessary to prove the following claims (in fact 13 terms are enough): f is a twist of $\Delta \in \mathcal{S}_{12}(1)$ and g is a twist of the unique newform in $\mathcal{S}_{12}(3)$ (both are twists by

χ_3). The factor x is only present in the coefficients a_n of h with $n \equiv 2 \pmod 3$ and hence $\sigma^-(a_n) = \chi_3(a_n)$, i.e. $h_{\chi_3}^+ = h^-$ meaning that h has an inner twist with χ_3 .

1.5. Road map. In Section 2 we will review the necessary background material and definitions. First we will recall basic facts about Fuchsian groups. In particular we will give detailed information about those congruence subgroups which play a role in Theorem 1.1. We will also introduce Maass waveforms, Hecke operators and the theory of Maass newforms. The various subspaces of Maass waveforms mentioned in the Main Theorem will then be properly defined. The section on background material ends with a precise formulation of the Selberg trace formula for the full modular group.

The next section is devoted to a more detailed study of Maass waveforms on the group Γ^3 . In particular we will give a complete description of the forms in this space which are associated to the modular group and the maps taking forms on Γ^3 to forms on $\Gamma_0(9)$. In Section 4 we give a precise definition of the genuinely new forms and a formulation and proof of the Main Theorem.

The most technical section of the paper is Section 5 which gives the details of the Selberg trace formula for all the subgroups we need. The section concludes with the proof of the key result, that the space of genuinely new forms on $\Gamma_0(9)$ is empty.

A slightly modified version of the Hecke theory for $\Gamma_0(9)$ is introduced in Section 6, for the purpose of demonstrating that the construction of multiplicities is compatible with Hecke operators (Proposition 1.4).

In the bulk of this paper we take a classical approach to Maass waveforms, that is, we treat them as functions on the complex upper half-plane. An alternative approach is given in terms of automorphic representations. In Section 7 we give an interpretation of our results in this setting. The last section contains a discussion on possible extensions of the results in this paper.

2. BACKGROUND MATERIAL

We will start with a brief recollection of basic facts concerning Fuchsian groups. For a more thorough treatment, the reader may consult any of a number of textbooks, for example [36, 9, 34] or the first chapter of [39], to name a few.

2.1. Hyperbolic geometry and Fuchsian groups. Let $\mathbb{H} = \{z = x + iy \mid y > 0\}$ be the hyperbolic upper half-plane equipped with metric and area measure $ds = y^{-1}|dz|$ and $d\mu = y^{-2}dx dy$, respectively. The boundary of \mathbb{H} is given by $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$ and a geodesic on \mathbb{H} is either a half-circle perpendicular to the real axis or a straight line parallel to the imaginary axis. The group of orientation preserving isometries of \mathbb{H} is $\mathrm{PSL}_2(\mathbb{R}) \simeq$

$\mathrm{SL}_2(\mathbb{R})/\{\pm \mathbb{1}_2\}$ acting on \mathbb{H} by Möbius transformations $\gamma: z \mapsto \gamma z = \frac{az+b}{cz+d}$. Here $\mathbb{1}_2$ is the two-by-two identity matrix. Composition of maps correspond to matrix multiplication and we will usually write elements of $\mathrm{PSL}_2(\mathbb{R})$ as matrices in $\mathrm{SL}_2(\mathbb{R})$. For notational clarity we will, whenever possible, use matrices with integer entries. For example, we will often use the following two maps in the remaining part of the paper:

$$(2.1) \quad \omega_d : z \mapsto -\frac{1}{dz} \quad \text{and}$$

$$(2.2) \quad V_d : z \mapsto dz.$$

Instead of matrices with determinant one, we will use the matrices $\begin{pmatrix} 0 & -1 \\ d & 0 \end{pmatrix}$ and $\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}$ to represent these maps. Since we will never deal with actual elements of matrix groups this should not cause any confusion, as long as it is understood that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}$ are equal if λ is a non-zero real number.

If $\varphi: \mathbb{H} \rightarrow \mathbb{C}$ is a function then $\mathrm{PSL}_2(\mathbb{R})$ acts on φ by the (weight zero) *slash-action*: $\gamma \mapsto \varphi|_\gamma$, where $\varphi|_\gamma(z) = \varphi(\gamma z)$. The Laplace-Beltrami operator on \mathbb{H} is given by $\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$ and since it only depends on the metric it commutes with the action of $\mathrm{PSL}_2(\mathbb{R})$ (this can also be checked explicitly). Analogous to the above action we can also define an action of $\mathrm{PGL}_2(\mathbb{R})$ on \mathbb{H} , setting $\gamma z = \frac{a\bar{z}+b}{c\bar{z}+d}$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $ad - bc = -1$. In practice, the only map of this form we will need is the reflection in the imaginary axis: $J: z \mapsto -\bar{z}$, which can be represented by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

A *Fuchsian group* is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$, by which we mean that if M_1, M_2, \dots is a sequence of elements in the group and $M_n \rightarrow \mathrm{Id}$, the identity in $\mathrm{PSL}_2(\mathbb{R})$, as $n \rightarrow \infty$, then there is an $M > 0$ such that $M_m = \mathrm{Id}$ for all $m \geq M$. The orbit space, or orbifold, given by the quotient of the upper half-plane by a Fuchsian group Γ , written $\Gamma \backslash \mathbb{H} := \{\Gamma z : z \in \mathbb{H}\}$ can be given a complex analytic structure which might be smooth or contain singularities, either in the form of corners (elliptic points), or of the form of punctures (parabolic points or cusps). It is common to represent $\Gamma \backslash \mathbb{H}$ in the upper half-plane by a fundamental domain, \mathcal{F}_Γ , which, for simplicity, we will always assume to be closed (i.e. parts of the boundary are counted twice), convex and bounded by a finite number of geodesic arcs (that is, we only consider Fuchsian groups which are geometrically finite). The elliptic points and cusps correspond to points z with non-trivial stabilizer subgroup: $\Gamma_z := \{\gamma \in \Gamma \mid \gamma z = z\}$. Such points necessarily appear on the boundary of \mathcal{F}_Γ and are inside \mathbb{H} (elliptic points) or on the boundary of \mathbb{H} (cusps). The stabilizer of an elliptic point is cyclic of finite order and the stabilizer of a parabolic point is infinite cyclic. The corresponding elements of Γ_z are also said to be elliptic or parabolic. The only remaining type of elements in $\mathrm{PSL}_2(\mathbb{R})$ are the so-called hyperbolic elements, possessing two different fixed points on $\partial \mathbb{H}$. It is easy to identify the type of a Möbius transformation from

the trace of its corresponding matrix: If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (and $ad - bc = 1$) then γ is elliptic, parabolic or hyperbolic if $|a + d|$ is less than, equal to, or greater than 2, respectively.

If \mathcal{F}_Γ is compact, meaning that there are no cusps, we say that Γ is co-compact and if the hyperbolic area of \mathcal{F}_Γ is finite we say that Γ is co-finite. All Fuchsian groups we will encounter in this paper are co-finite but not co-compact.

If $\Gamma' \subset \Gamma$ is a subgroup of finite index we can obtain a fundamental domain of Γ' by using right coset-representatives, i.e. if $\Gamma = \sqcup_{i=1}^\mu \Gamma' R_i$ then $\mathcal{F}_{\Gamma'} = \sqcup_{i=1}^\mu R_i \mathcal{F}_\Gamma$ is a fundamental domain for Γ' . Here the symbol \sqcup is understood to mean a disjoint union of (discrete) sets in the first expression and a union of subsets of \mathbb{R}^2 with pair-wise intersections having Lebesgue measure zero in the second expression.

2.2. Examples of Fuchsian groups. An elementary but important family of Fuchsian groups are the *Hecke triangle groups*, G_n , generated by a reflection $S : z \mapsto -\frac{1}{z}$ and a translation $T_\lambda : z \mapsto z + \lambda$, where $\lambda = \lambda_n := 2 \cos \frac{\pi}{n}$ and $n \geq 3$ is an integer (for other values of λ the resulting group will not be Fuchsian). The standard fundamental domain, $\mathcal{F}_n := \mathcal{F}_{G_n}$ of the Hecke triangle group G_n can be taken as the hyperbolic triangle with one vertex at infinity (a cusp) and the other two vertices at the intersection of the unit circle with the vertical lines $x = \pm \frac{\lambda}{2}$ in the upper half-plane (these are elliptic points equivalent under the group). In fact \mathcal{F}_n also has one additional elliptic point at i , meaning that \mathcal{F}_n is in fact a rectangle, but with one angle equal to π .

A special case of a Hecke triangle group and the canonical example of a Fuchsian group in number theory is the *modular group*, $\mathrm{PSL}_2(\mathbb{Z})$ ($= G_3$) which has a simple presentation in terms of the generators $S : z \mapsto -\frac{1}{z}$ and $T : z \mapsto z + 1$ and relations $S^2 = (ST)^3 = \mathrm{Id}$. The corresponding matrices are $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

For any integer $N \geq 1$ there is a projection $\pi_N : \mathbb{Z} \rightarrow \mathbb{Z}/N\mathbb{Z}$, $n \mapsto n \bmod N$, which induces a surjective group homomorphism $\pi_N : \mathrm{PSL}_2(\mathbb{Z}) \rightarrow \mathrm{PSL}_2(\mathbb{Z}/N\mathbb{Z})$. Its kernel, $\Gamma(N) = \{\gamma \in \mathrm{PSL}_2(\mathbb{Z}) \mid \gamma \equiv \pm \mathbb{1}_2 \bmod N\}$, is called the principal congruence subgroup of level N . A subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ which contains $\Gamma(N)$ as a subgroup, but no $\Gamma(M)$ for $M < N$ is said to be a *congruence subgroup* of level N . We are interested in the *Hecke congruence subgroup* of level N , $\Gamma_0(N) = \{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) \mid c \equiv 0 \bmod N\}$. The index of $\Gamma(N)$ and $\Gamma_0(N)$ in $\mathrm{PSL}_2(\mathbb{Z})$ is given by $\mu(N) = \frac{1}{2}N^3 \prod_{p|N} (1 - p^{-2})$ and $\mu_0(N) := N \prod_{p|N} (1 + p^{-1})$, respectively, where the products are taken over all primes $p|N$ (see for example [50, Thm. 1.4.1 and (1.4.24)]).

For a prime level, p , it is easy to verify directly that the index of $\Gamma_0(p)$ in $\mathrm{PSL}_2(\mathbb{Z})$ is $p + 1$ and that $R_0 = \mathrm{Id}, R_1 = S, R_2 = ST, \dots, R_p = ST^{p-1}$ is a set of $p + 1$ maps which are independent modulo $\Gamma_0(p)$ and therefore constitute a set of right coset representatives for $\Gamma_0(p) \backslash \mathrm{PSL}_2(\mathbb{Z})$. For general non-prime levels it is hard to find similar simple expressions.

The best approach is then to use the generators S and T to construct μ_N independent elements. This approach is also suitable for implementation on a computer.

For the purpose of this paper we need to study the subgroups $\Gamma_0(3)$, $\Gamma_0(9)$, $\Gamma(3)$ and Γ^3 more closely. The group Γ^3 (which we will define in Example 2.3 below) is an example of what is called a *cycloidal* subgroup of the modular group, i.e. a group with only one cusp.

Example 2.1. The index of $\Gamma_0(9)$ in $\mathrm{PSL}_2(\mathbb{Z})$ is $\mu(9) = 12$. Hence it is clear that the set $\{R_1, \dots, R_{10}\}$ with $R_1 = \mathrm{Id}$ and $R_j = ST^{j-2}$ for $j = 2, \dots, 10$ is not sufficient as coset representatives for $\Gamma_0(9) \backslash \mathrm{PSL}_2(\mathbb{Z})$. It is however easy to verify that the two maps $R_{11} = ST^{-3}S = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ and $R_{12} = ST^3S = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$ are independent modulo $\Gamma_0(9)$, both of each other and of the set $\{R_1, \dots, R_{10}\}$.

The index of $\Gamma_0(9)$ in $\Gamma_0(3)$ is 3 and the maps $P_1 = \mathrm{Id}$, $P_2 = ST^3S$ and $P_3 = P_2^2 = ST^6S$ satisfy $P_2, P_3 \in \Gamma_0(3) \backslash \Gamma_0(9)$. Since $P_3 P_2^{-1} = P_2$ and $P_2 P_3^{-1} = P_2^{-1}$ these maps are also independent modulo $\Gamma_0(9)$. We have thus shown that the following set of maps is suitable as coset representatives:

$$\Gamma_0(9) \backslash \mathrm{PSL}_2(\mathbb{Z}) \simeq \{R_1, \dots, R_{12}\} \quad \text{and} \quad \Gamma_0(9) \backslash \Gamma_0(3) \simeq \{P_1, P_2, P_3\}.$$

Another fact which we will use later is that $\Gamma_0(9)$ is conjugate to $\Gamma(3)$: If $\gamma = \begin{pmatrix} a & b \\ 9c & d \end{pmatrix} \in \Gamma_0(9)$ then $V_3 \gamma V_3^{-1} = \begin{pmatrix} a & 3b \\ 3c & d \end{pmatrix}$ and since $ad \equiv 1 \pmod{3}$ implies that $a \equiv d \equiv \pm 1 \pmod{3}$ it follows that $V_3 \Gamma_0(9) V_3^{-1} = \Gamma(3)$.

Example 2.2. Since the index of $\Gamma(3)$ in $\mathrm{PSL}_2(\mathbb{Z})$ is 12 it follows that we need 3 coset representatives for $\Gamma(3) \backslash \Gamma_0(3)$. A short calculation shows that we can take $R_1 = \mathrm{Id}$, $R_2 = T$ and $R_3 = T^2$ as coset representatives, i.e. $\Gamma(3) \backslash \Gamma_0(3) \simeq \{\mathrm{Id}, T, T^2\}$.

Example 2.3. Let $h : \mathrm{PSL}_2(\mathbb{Z}) \rightarrow \mathbb{Z}/3\mathbb{Z}$ be defined by $h(A) = a_0 + a_1 + \dots + a_n \pmod{3}$ if $A = T^{a_0} S T^{a_1} \dots S T^{a_n}$. Although this representation of A is not unique it is well-known that the only relations in $\mathrm{PSL}_2(\mathbb{Z})$ are $S^2 = (ST)^3 = \mathrm{Id}$. Therefore the function h is well-defined and in fact a surjective group homomorphism. Its kernel,

$$(2.3) \quad \Gamma^3 = \{A \in \mathrm{PSL}_2(\mathbb{Z}) \mid h(A) = 0\},$$

is therefore an index 3 subgroup of $\mathrm{PSL}_2(\mathbb{Z})$, with coset representatives $\{\mathrm{Id}, T, T^2\}$. By using the free generators S and $R = ST$ instead of S and T to define the function h it is not hard to show that S together with A_1 and A_2 generate Γ^3 , where $A_1 = R^{-1}SR = T^{-1}ST = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$, $A_2 = R^{-2}SR^2 = ST^2ST = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}$ and $A_1^2 = A_2^2 = \mathrm{Id}$. The following alternative description is sometimes useful:

$$(2.4) \quad \Gamma^3 = \{A \in \mathrm{PSL}_2(\mathbb{Z}), \pm A \equiv \mathrm{Id}, S, A_1 \text{ or } A_2 \pmod{3}\}.$$

Let $\tilde{\Gamma}$ denote the right hand side. Since the generators of Γ^3 is in $\tilde{\Gamma}$ we have $\Gamma^3 \subseteq \tilde{\Gamma}$. It is easy to verify that any element A of $\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$ can be written as a product $A = BT^m$ where B is congruent modulo 3 to one of the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$

$\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$, $\begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ (corresponding to $\tilde{\Gamma}$) and $m = 0, 1$ or 2 . It follows that $\tilde{\Gamma}$ is a congruence subgroup of level 3 and has index 3 in the modular group. Since Γ^3 also has index 3 it follows that $\Gamma^3 = \tilde{\Gamma}$. In particular we see that Γ^3 is indeed a congruence subgroup of level 3. The following description can be deduced from (2.4):

$$(2.5) \quad \Gamma^3 = \{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) \mid ab + cd \equiv 0 \pmod{3}\}.$$

The group Γ^3 and other (similarly defined) groups were studied in detail by Rankin [50, ch. 1] (see also Fricke [20, p. 532]).

2.3. Normalizers of Fuchsian groups. Let Γ be a co-finite Fuchsian group. The *normalizer* of Γ in $\mathrm{PSL}_2(\mathbb{R})$, denoted $\mathrm{Norm}(\Gamma)$, is the group consisting of all elements γ in $\mathrm{PSL}_2(\mathbb{R})$ which leaves Γ invariant under conjugation, i.e. $\Gamma = \gamma\Gamma\gamma^{-1}$. In certain cases it is not so hard to study the normalizer in a geometric manner, using the concept of maximality. We say that Γ is maximal if there does not exist any other Fuchsian group G containing Γ as a subgroup of finite index (this is sometimes called finitely maximal in the literature).

From the relationship between subgroups and fundamental domains, as expressed in the last paragraph of Section 2.1 we see that a geometric method to find out if a given group F is maximal or not is to study tessellations of a fundamental domain for F into equally sized parts (with respect to hyperbolic area measure) identified by elements of $\mathrm{PSL}_2(\mathbb{R})$ (recall that these preserve angles and orientation).

The simplest example of a non-co-compact fundamental domain is the standard domain of a of Hecke triangle group G_q , which is, as described above, simply a triangle with one angle equal to zero and the other two angles equal to $\frac{\pi}{q}$. Since any supergroup of G_q must have precisely one cusp it is immediately clear that the only possible partition of the fundamental domain is to split it along the imaginary axis. However, the identification of these parts is by the reflection $J : z \mapsto -\bar{z}$ which belongs to $\mathrm{PGL}_2(\mathbb{R})$ but not $\mathrm{PSL}_2(\mathbb{R})$. It follows that all Hecke triangle groups are maximal. For this and related results see Greenberg [23] and Beardon [8].

If $\omega \in \mathrm{Norm}(\Gamma)$ and we let $\Gamma^* = \langle \Gamma, \omega \rangle$ be the group generated by the elements of Γ together with ω then Γ^* contains Γ as a subgroup. It follows that a maximal Fuchsian group must have a trivial normalizer and to find the normalizer of a given group it is therefore enough to find a minimal set of elements which we can use to extend Γ to a maximal Fuchsian group.

For $\Gamma_0(p)$ with $p = 2$ or 3 it is straight-forward to check that $\mathrm{Norm}(\Gamma_0(p)) = \Gamma_0^*(p)$ where $\Gamma_0^*(p) = \Gamma_0(p) \sqcup \Gamma_0(p)\omega_p$ with $\omega_p : z \mapsto -\frac{1}{pz}$ the so-called Fricke involution (cf. [20, p. 357]): simply observe that the dilation $V_{\sqrt{p}} : z \mapsto \sqrt{p}z$ conjugates $\Gamma_0^*(p)$ to $G_{\sqrt{p}}$, which is maximal. It is in fact true for all primes p that

$$\mathrm{Norm}(\Gamma_0(p))/\Gamma_0(p) = \langle \omega_p \rangle.$$

Note that $\omega_p^2 = \text{Id}$. Since the group $\Gamma_0^*(p)$ is in general not a triangle group the corresponding maximality argument becomes more involved. The signature of $\Gamma_0^*(p)$ can be obtained from [24] (see also [20, pp. 363,366]) and we could then try to match this signature with the list of signatures corresponding to maximal Fuchsian groups in [56]. The most general formulas for the normalizer of $\Gamma_0(N)$ (general N) are proved using a more algebraic approach with matrix calculations. A general formula for the elements of the normalizer is given explicitly by Lehner and Newman [37] and in a slightly more comprehensive form by Atkin and Lehner [3]. It turns out that for each prime Q dividing N the normalizer contains an involution, W_Q , of the form $W_Q = \begin{pmatrix} Qx & y \\ Nz & Qw \end{pmatrix}$ with $x, y, z, w \in \mathbb{Z}$, $x \equiv 1 \pmod{\frac{N}{Q}}$, $y \equiv 1 \pmod{Q}$ and $Q^2xw - Nyz = Q$. These are usually called Atkin-Lehner involutions. Although the latter papers contain correct formulas for the elements of the normalizer, the stated formulas for the group structure of $\text{Norm}(\Gamma_0(N))/\Gamma_0(N)$ contains minor errors in some cases when N is divisible by either 4 or 9. The correct group structure can be found in e.g. Akbas and Singerman [1] or Bars [7].

For the most important group in this paper, $\Gamma_0(9)$, the situation is slightly different. Consider the maps ω_9 and $T^{1/3}$. These are in the normalizer of $\Gamma_0(9)$ and are given by

$$\omega_9 : z \mapsto -\frac{1}{9z}, \quad \text{and} \quad T^{1/3} : z \mapsto z + \frac{1}{3}.$$

Define $\Gamma_0^*(9) = \Gamma_0(9) \sqcup \Gamma_0(9)\omega_9 \sqcup \Gamma_0(9)T^{1/3}$. We know that $\Gamma_0(9)$ has four inequivalent cusps, which can be represented by $0, -\frac{1}{3}, \frac{1}{3}$ and ∞ (cf. e.g. [31, Prop. 2.6]). Since $\omega_9(0) = \omega_9 T^{1/3}(\frac{-1}{3}) = \omega_9 T^{-1/3}(\frac{1}{3}) = \infty$ it is clear that $\Gamma_0^*(9)$ has only one cusp. It is also readily verified that (recall that $V_3 z = 3z$) $V_3 \omega_9 V_3^{-1} = S$, $V_3 T^{1/3} V_3^{-1} = T$ and $V_3 \gamma V_3^{-1} \in \Gamma_0(3)$ for any $\gamma \in \Gamma_0(9)$. That is, $V_3 \Gamma_0^*(9) V_3^{-1} = \text{PSL}_2(\mathbb{Z})$, and in particular, it follows that $\Gamma_0^*(9)$ is maximal. Since $(T^{1/3})^3 = T \in \Gamma_0(9)$, $\omega_9^2 = \text{Id}$ and $(\omega_9 T^{1/3})^3 = \text{Id}$ we see that the normalizer of $\Gamma_0(9)$ modulo $\Gamma_0(9)$ is isomorphic to a group with the following presentation:

$$\text{Norm}(\Gamma_0(9))/\Gamma_0(9) \simeq \langle x, y \mid x^2 = y^3 = (xy)^3 = 1 \rangle$$

From the definition (2.3) we see that T is in the normalizer of Γ^3 . Since $S \in \Gamma^3$ it follows that $\text{Norm}(\Gamma^3) = \text{PSL}_2(\mathbb{Z})$ and hence

$$\text{Norm}(\Gamma^3)/\Gamma^3 \simeq \langle x \mid x^3 = 1 \rangle$$

It is also possible to consider the normalizer of $\Gamma_0(N)$ in $\text{PGL}_2(\mathbb{R})$ instead of $\text{PSL}_2(\mathbb{R})$. It is not hard to verify that $J : z \mapsto -\bar{z}$ is an involution of all $\Gamma_0(N)$ and that in fact the normalizer of $\Gamma_0(N)$ in $\text{PGL}_2(\mathbb{R})$ is the group generated by $\text{Norm}(\Gamma_0(N))$ together with J .

Observe that J commutes with the Fricke involutions ω_N but $JTJ = T^{-1}$ so the normalizer of Γ^3 in $\mathrm{PGL}_2(\mathbb{R})$ modulo Γ^3 has the presentation $\langle x, y \mid x^3 = y^2 = yxyx = 1 \rangle$.

2.4. Maass waveforms. Let $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{Z})$ be a congruence subgroup of level N . It is well-known (cf. e.g. [32]) that the discrete spectrum of Δ on $\Gamma \backslash \mathbb{H}$ is spanned by *Maass waveforms*, that is, real-analytic eigenfunctions of Δ on \mathbb{H} , which are invariant under Γ . Furthermore, in this case we know that these forms are cuspidal, i.e. vanish at the cusps of Γ , and have finite L^2 -norm induced by the Petersson inner product:

$$(2.6) \quad \langle f, g \rangle_\Gamma = \int_{\mathcal{F}_\Gamma} f(z) \overline{g(z)} d\mu.$$

The integral above can be taken over any closed fundamental domain \mathcal{F}_Γ of Γ . *It is tacitly understood that, in the remainder of the paper, all mentioned Maass waveforms are cusp forms.* The space of all Maass waveforms on Γ with Laplace eigenvalue $\lambda = \frac{1}{4} + R^2$ is denoted by $\mathcal{M}(\Gamma, R)$ and in the case $\Gamma = \Gamma_0(N)$ we also write $\mathcal{M}(N, R)$.

Since the Laplacian commutes with the action of $\mathrm{PGL}_2(\mathbb{R})$ it is easy to check that all operators (acting on Maass waveforms) which we consider in this paper preserve the Laplace eigenvalue. *We will therefore sometimes let $\mathcal{M}(\Gamma)$ or $\mathcal{M}(N)$ denote a generic non-empty eigenspace (and omit the spectral parameter R).* The theory of Hecke operators on holomorphic modular forms carry over with only minor changes to Maass waveforms. See e.g. [62, III].

Definition 2.4. For integers $n, N \geq 1$ we define the Hecke operator T_n of level N by

$$T_n \varphi(z) = \frac{1}{\sqrt{n}} \sum_{\substack{ad=n \\ (a,N)=1, a>0}} \sum_{b \bmod d} \varphi\left(\frac{az+b}{d}\right) \quad \text{for any } \varphi \in \mathcal{M}(N).$$

Note that $T_1 = \mathrm{Id}$ and with the analogous definition for $n = -1$ we get $T_{-1} = J$. The operators T_n satisfy all the properties familiar from the holomorphic theory. In particular, the set of operators T_n with $(n, N) = 1$ is a family of commuting, self-adjoint operators satisfying the following relation for $m, n \in \mathbb{Z}^+$:

$$T_m T_n = \sum_{\substack{d \mid (m,n) \\ (d,N)=1, d>0}} \frac{T_{mn}}{d^2}.$$

Having a Hecke theory, we will now discuss the space of oldforms and its orthogonal complement the space of newforms, in the usual sense of Atkin-Lehner [3].

A fundamental lemma in the theory of oldforms is that if M and N are positive integers with $N = Md$ with $d > 1$ then there are two distinguished ways to map a function $\varphi \in \mathcal{M}(M)$ to a function $\tilde{\varphi} \in \mathcal{M}(N)$. Since $\Gamma_0(N) \subset \Gamma_0(M)$ and it follows that $\varphi \in \mathcal{M}(N)$. In addition it is easy to verify that $V_d \Gamma_0(N) V_d^{-1} \subseteq \Gamma_0(M)$ and therefore $\varphi|_{V_d}(z) = \varphi(dz)$ also satisfies

$\varphi|_{V_d} \in \mathcal{M}(N)$. The space of old Maass waveforms, or simply *oldforms*, on $\Gamma_0(N)$, denoted by $\mathcal{M}^o(N) = \bigoplus_R \mathcal{M}^o(N, R)$, is defined as the linear span of all forms of the type $f(z)$ and $f(dz)$ with $f \in \mathcal{M}(N/d, R)$, $d|N$ and $d \neq 1$. The *new space*, $\mathcal{M}^n(N)$, is then defined as the orthogonal complement of the old space, with respect to the Petersson inner product. The most important property of the decomposition into the old and new subspace is that both spaces are invariant under the Hecke operators T_n with $(n, N) = 1$. The following definition is then the direct analogue of the classical definition in [3] for holomorphic forms.

Definition 2.5. By a (normalized) *Maass newform* on $\Gamma_0(N)$ we mean a $\varphi \in \mathcal{M}^n(N, R)$ which is an eigenfunction of all Hecke operators T_n with $(n, N) = 1$ and of the reflection J . Additionally, we assume it is normalized to have first Fourier coefficient at infinity equal to 1.

It is well-known that each $\mathcal{M}^n(N, R)$ has an orthogonal (finite) basis of Maass newforms (cf. e.g. [62, Thm. 4.6, p. 94]). By a direct calculation (or using [3, lemma 17]) we see that the Hecke operators T_n with $(n, N) = 1$, together with the Fricke involution ω_N and the other Atkin-Lehner involutions W_Q , with Q dividing N , form a commuting family of normal, linear operators on $\mathcal{M}^n(N, R)$.

Since $J \begin{pmatrix} a & b \\ c & d \end{pmatrix} J^{-1} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$ it follows that J commutes with the Hecke operators and the Fricke involution but $JW_QJ^{-1} = W_{-Q}$ and $JTJ^{-1} = T^{-1}$. If we want to desymmetrize the space $\mathcal{M}(N)$ as much as possible we therefore might have to make a choice: either include all Atkin-Lehner involutions, or only some of them, together with J . For $\Gamma_0(9)$ we also have the problem that $T^{1/3}$ neither commutes with J nor all Hecke operators. The Hecke theory for $\Gamma_0(9)$ will be studied further in Section 6.

Analogous to the above definition for groups of the type $\Gamma_0(N)$ we also define the space of oldforms on Γ^3 , denoted $\mathcal{M}^o(\Gamma^3)$ as the subspace of $\mathcal{M}(\Gamma^3)$ spanned by “lifts” of forms from $\mathcal{M}(1)$. As we will see in Corollary 3.3 the only way to achieve such a form is the trivial one (by inclusion of Γ^3 in $\text{PSL}_2(\mathbb{Z})$). The space of newforms on Γ^3 , $\mathcal{M}^n(\Gamma^3)$ is then defined as the orthogonal complement of the space of oldforms. To avoid introducing Hecke operators on Γ^3 we simply use the term newform for any element of $\mathcal{M}^n(\Gamma^3)$.

In analogy with the holomorphic case we say that a Maass form f is of CM-type if it possesses a self-twist, that is, if there exists a Dirichlet character χ such that $f_\chi = f$. For a more precise discussion of CM-forms with more references see e.g. [11, Ch. 3] and [29]. The only thing we need to know about CM-forms is that there are very few compared to the remaining forms of “general type” (in fact there are none on $\Gamma_0(9)$).

2.5. Twists and forms associated to other groups. The action of the Hecke operators (and the fact that an infinite subset of these form a commutative algebra with special multiplicative properties) is one of the most important tools in the study of automorphic forms

on congruence subgroups. It is therefore natural to require that any systematic way of obtaining automorphic forms on one congruence subgroup from those on another should be compatible with this action. There are two prominent methods to do this: *level raising* (i.e. producing oldforms by the inclusion or by the map V_d) and *twisting*.

Suppose that Γ and Γ' are two congruence subgroups, $\varphi \in \mathcal{M}(\Gamma)$ and $A \in \mathrm{PGL}_2(\mathbb{Q})$ is such that $A\Gamma'A^{-1} \subseteq \Gamma$. Then $\varphi|_A \in \mathcal{M}(\Gamma')$. It is also easy to verify that if, for any $\alpha \in \mathrm{PSL}_2(\mathbb{Q})$, $\Gamma\alpha\Gamma$ is commensurable with Γ (i.e. $\Gamma\alpha\Gamma \cap \Gamma$ has finite index in both $\Gamma\alpha\Gamma$ and Γ) then $\Gamma'\tilde{\alpha}\Gamma'$ is commensurable with Γ' , where $\tilde{\alpha} = A^{-1}\alpha A \in \mathrm{PSL}_2(\mathbb{Q})$. Thus, with the more general definition of Hecke operators as double coset operators as in e.g. Miyake [39, ch. 5] or Shimura [55] it is clear that this type of correspondence is compatible with the Hecke operators. Note that, however, the conjugated operator $A^{-1}T_n A$ on Γ' might be different from T_n on Γ .

We have already seen one example of the previous construction: In the definition of oldforms we had $\Gamma = \Gamma_0(M)$, $\Gamma' = \Gamma_0(Md)$ and $A = V_d$. We will use the following example later.

Example 2.6. From Example 2.1 we know that $V_3\Gamma_0(9)V_3^{-1} = \Gamma(3) \subseteq \Gamma^3$ and hence V_3 provides a map from $\mathcal{M}(\Gamma^3)$ to $\mathcal{M}(\Gamma_0(9))$.

If m is a positive integer, a Dirichlet character χ modulo m is a multiplicative function of period m with $\chi(n) = 0$ if $(m, n) > 1$ and $\chi(1) = 1$. The smallest period of χ is called the conductor of χ .

Definition 2.7. If $\varphi \in \mathcal{M}(N)$ and χ is a Dirichlet character of conductor q we define the *twist of φ by χ* as

$$\varphi_\chi(z) = \tau(\overline{\chi})^{-1} \sum_{n \bmod q} \overline{\chi}(n) \varphi\left(z + \frac{n}{q}\right)$$

where $\tau(\chi) = \sum_{n \bmod q} \chi(n) e\left(\frac{n}{q}\right)$ is the usual Gauss sum (cf. e.g. Miyake [39, Cor. 3.3.2.]).

An alternative definition of φ_χ is that if φ has Fourier coefficients a_n with respect to the cusp at infinity, then φ_χ has coefficients $\chi(n)a_n$ at the same cusp. It is straight-forward to verify that [39, Lemma 4.3.10] also holds for Maass waveforms. From this lemma we first of all conclude that the two definitions are equivalent. Second, we also conclude that if $\varphi \in \mathcal{M}^n(N)$ then $\varphi_\chi \in \mathcal{M}(M)$ with $M = \mathrm{lcm}(N, q^2)$.

The advantage of working with the first definition is that it is an explicit formula which relates twisting by $\chi_3 = \left(\frac{\cdot}{3}\right)$ in a natural way to the map $T^{1/3}$ in the normalizer of $\Gamma_0(9)$. It is otherwise more common to work with the definition in terms of Fourier coefficients.

In this case, if φ is a Hecke eigenform with eigenvalues λ_p then φ_χ is a Hecke eigenform with eigenvalues $\chi(p)\lambda_p$ for $(p, N) = (p, q) = 1$.

Using the definition above, an elementary matrix calculation shows directly that φ_χ is invariant under the group $\Gamma_0(M)$. However, to determine the exact level of φ_χ for a general starting level seems to be hard using only elementary methods. In Lemma 3.8 we will see that in the cases we need, if φ is a newform, then φ_χ is in fact a newform on $\Gamma_0(9)$.

2.6. The Selberg trace formula for the modular group. Let Γ be a co-finite Fuchsian group and let $0 = \lambda_0 < \lambda_1 \leq \dots$ denote the discrete spectrum of Γ (counted with multiplicity). We also write $\lambda_n = \frac{1}{4} + r_n^2$ with $r_n \in \mathbb{R} \cup i[0, \frac{1}{2}]$ and set $\sigma(\Gamma) = \{r_n \mid n \in \mathbb{N}_0\}$. Let $h(r)$ be an even analytic function and let $g(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) e^{-iur} dr$ be the Fourier transform of h . If there exists a $\delta > 0$ such that $h = O(|r|^{-(2+\delta)})$ in the strip $|\Im(r)| \leq \frac{1}{2} + \delta$ as $\Re(r) \rightarrow \infty$ then the *Selberg trace formula* (cf. e.g. [25, 26]) says that

$$(2.7) \quad \sum_{r_n \in \sigma(\Gamma)} h(r_n) = I(\Gamma) + E(\Gamma) + H(\Gamma) + P(\Gamma)$$

where I, E, H, P denotes the contributions of the identity, elliptic, hyperbolic and parabolic conjugacy classes in Γ . These terms all depend on h but to simplify the notation we will omit this dependence. By using a suitable test function it is possible to prove the analogue of Weyl's law for the (possibly non-compact) orbifold $\Gamma \backslash \mathbb{H}$:

$$(2.8) \quad N_\Gamma(T) - M_\Gamma(T) = \frac{\text{vol}(\mathcal{F}_\Gamma)}{4\pi} T^2 + O(T \ln T), \quad \text{as } T \rightarrow \infty.$$

Here $N_\Gamma(T) = |\{r_n \in \sigma(\Gamma), r_n \leq T\}|$ is the counting function for the discrete spectrum and $M_\Gamma(T)$ is essentially a counting function for the continuous spectrum, related to the winding number for φ_Γ , the scattering determinant of Γ . For a generic (non-arithmetic) Fuchsian group it is believed (cf. e.g. Phillips-Sarnak [42, 43]) that M_Γ dominates and that the discrete spectrum is at most finite. For cycloidal and congruence subgroups of the modular group it is possible to show that $M_\Gamma(T) = O(T \ln T)$ as $T \rightarrow \infty$ and therefore there exists an infinite number of discrete eigenvalues in these cases. For an overview of Weyl's law in different settings see e.g. [2]. For the case of Fuchsian groups in general and congruence congruence groups in particular see also e.g. [26, Thm. 2.28 and Ch. 11], Venkov [65] and Risager [53]. To compute the right hand side of Weyl's law (2.8) when Γ is a subgroup of $\text{SL}_2(\mathbb{Z})$ of index μ we recall that $\text{vol}(\mathcal{F}_{\text{SL}_2(\mathbb{Z})}) = \frac{\pi}{3}$ so that the main term is $\frac{\mu}{12} T^2$.

We will now give the explicit form of the terms above in the case of the modular group. For simplicity we denote the corresponding terms by I_1, E_1, H_1 and P_1 respectively. We also write $E_1 = E_1(2) + E_1(3)$ where $E_1(m)$ denotes the contribution from all elliptic classes of

order m . By using e.g. [26, p. 209] we obtain

$$(2.9) \quad \begin{aligned} I_1 &= \frac{1}{12} \int_{-\infty}^{\infty} r h(r) \tanh \pi r dr, \\ E_1(2) &= \frac{1}{4} \int_0^{\infty} h(r) \frac{1}{\cosh \pi r} dr, \\ E_1(3) &= \frac{2}{3\sqrt{3}} \int_0^{\infty} h(r) \frac{\cosh \frac{\pi r}{3}}{\cosh \pi r} dr \quad \text{and} \end{aligned}$$

$$(2.10) \quad \begin{aligned} P_1 &= \frac{1}{4} h(0) \left[1 - \varphi_1 \left(\frac{1}{2} \right) \right] - g(0) \ln 2 \\ &\quad + \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) \frac{\varphi_1'}{\varphi_1} \left(\frac{1}{2} + ir \right) dr - \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) \frac{\Gamma'(1+ir)}{\Gamma(1+ir)} dr \end{aligned}$$

where

$$(2.11) \quad \varphi_1(s) := \varphi_{\text{PSL}_2(\mathbb{Z})}(s) = \frac{\Lambda(2-2s)}{\Lambda(2s)}$$

and $\Lambda(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$ is the completed Riemann zeta function. Although the precise form of the hyperbolic term is not important for the proof later we will describe it anyway, for the sake of completeness. A hyperbolic element $P \in \text{PSL}_2(\mathbb{Z})$ has two different fixed points on $\mathbb{R} \cup \{\infty\}$ and acts as a dilation on the geodesic γ_P which connects them. That is, P is conjugate in $\text{PSL}_2(\mathbb{R})$ to a map of the form $V_{N(P)} : z \mapsto N(P)z$ (by virtue of the map which takes one fixed point to 0 and the other to ∞). The number $N(P)$ is sometimes called the norm of P and $\ln N(P)$ is the (hyperbolic) length of the closed geodesic on $\text{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}$ given by the projection of γ_P . We use P_0 to denote the hyperbolic element of smallest norm such that $P = P_0^m$ for some $m \in \mathbb{N}$. If $P = P_0$ we say that P is primitive. It is now clear that there is a bijection between hyperbolic conjugacy classes and closed geodesics on $\text{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}$. The hyperbolic contribution to the Selberg trace formula can now be written as

$$H_1 = \sum_{\{P\}} \frac{\ln N(P_0)}{N(P)^{\frac{1}{2}} - N(P)^{-\frac{1}{2}}} g(\ln N(P))$$

where the sum runs over all hyperbolic conjugacy classes in $\text{PSL}_2(\mathbb{Z})$. By collecting terms corresponding to the same length we can write

$$H_1 = \sum_{n=0}^{\infty} c_n g(x_n)$$

where $0 < x_0 < x_1 < \dots \rightarrow \infty$ is the *length spectrum* of the modular surface, i.e. the sequence of (distinct) lengths of its geodesics, and $c_n > 0$ are constants. Note that we sum over *all* lengths, and not only the primitive ones. The length spectrum of a subgroup is

clearly contained in that of the larger group. Hence, if Γ is any finite index subgroup of the modular group then

$$H(\Gamma) = \sum_{n=0}^{\infty} c_n(\Gamma) g(x_n)$$

with the same lengths x_n and some other non-negative constants $c_n(\Gamma)$. In section 5 we will see how the other terms in the Selberg trace formula for subgroups can be expressed in terms of those for the modular group.

3. MAASS CUSP FORMS ON Γ^3 AND $\Gamma_0(9)$

In this section we will present technical lemmas about the precise nature of the maps between spaces of Maass waveforms introduced in Section 2.5. First we will discuss maps from $\mathrm{PSL}_2(\mathbb{Z})$ into Γ^3 and from Γ^3 into $\Gamma_0(9)$. Then we will discuss twisting of forms on $\mathrm{PSL}_2(\mathbb{Z})$ and $\Gamma_0(3)$ into forms on $\Gamma_0(9)$.

3.1. Maass waveforms on Γ^3 . From Section 2.3 we know that the normalizer of Γ^3 has two elements $J : z \mapsto -\bar{z}$ and $T : z \mapsto z + 1$, satisfying $J^2 = T^3 = JTJT = \mathrm{Id}$ (modulo Γ^3). Using (2.6) and a change of variables $z \mapsto z + 1$ in the integral, together with the fact that the quotient $\Gamma^3 \backslash \mathbb{H}$ is invariant under this map, it is easy to check that $T : f \mapsto f|_T$ is a unitary operator of order three on $\mathcal{M}(\Gamma^3, R)$. We hence obtain an orthogonal decomposition:

$$(3.1) \quad \mathcal{M}(\Gamma^3, R) = \mathcal{M}(\Gamma^3, R)^{(0)} \oplus \mathcal{M}(\Gamma^3, R)^{(1)} \oplus \mathcal{M}(\Gamma^3, R)^{(-1)}$$

corresponding to the eigenvalues $1, \zeta_3, \zeta_3^{-1}$ of T on this space. Here $\zeta_3 := e^{\frac{2\pi i}{3}}$. Since $\mathrm{PSL}_2(\mathbb{Z})$ is generated by $S \in \Gamma^3$ together with T we see immediately that

$$\mathcal{M}^0(\Gamma^3, R) = \mathcal{M}(\Gamma^3, R)^{(0)} \quad \text{and} \quad \mathcal{M}^n(\Gamma^3, R) = \mathcal{M}(\Gamma^3, R)^{(1)} \oplus \mathcal{M}(\Gamma^3, R)^{(-1)}.$$

Since $JTJ = T^{-1}$ it follows that if $\varphi \in \mathcal{M}(\Gamma^3, R)^{(1)}$ then $\varphi|_J \in \mathcal{M}(\Gamma^3, R)^{(-1)}$ (and vice versa) and we immediately deduce the following lemma.

Lemma 3.1. *If the space $\mathcal{M}^n(\Gamma^3, R)$ is non-empty, then its dimension is at least two.*

3.2. Maps into $\mathcal{M}(\Gamma^3)$. An important part of the proof of the Main Theorem is the existence of newforms on Γ^3 (cf. last paragraph of Section 2.5). The precise statement of the existence is given in Proposition 3.4 below and for the proof we need the next two lemmas.

Lemma 3.2. *If $M \in \mathrm{PGL}_2(\mathbb{Q})$ is such that $MAM^{-1} \in \mathrm{PSL}_2(\mathbb{Z})$ for every $A \in \Gamma^3$ then $M \in \mathrm{PSL}_2(\mathbb{Z}) \cup \mathrm{PSL}_2(\mathbb{Z})J$.*

Proof. It is well-known that any $M \in \mathrm{PGL}_2(\mathbb{Q})$ can be represented by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a, b, c, d are integers, $\gcd(a, b, c, d) = 1$ and the determinant is $ad - bc = r \neq 0$. It turns out that it is enough to consider the action of M on the matrices S and T^3 from Γ^3 :

$$MT^3M^{-1} = \frac{1}{r} \begin{pmatrix} r-3ac & 3a^2 \\ -3c^2 & r+3ac \end{pmatrix} \quad \text{and} \quad MSM^{-1} = \frac{1}{r} \begin{pmatrix} ac+bd & -a^2-b^2 \\ c^2+d^2 & -ac-bd \end{pmatrix}.$$

Assume that MT^3M^{-1} and MSM^{-1} are both elements of $\mathrm{PSL}_2(\mathbb{Z})$ and that p is a prime dividing r . If $p \neq 3$ it is immediate that p divides a and c from the first matrix and that then p must divide b and d as well, from the second matrix. Hence $p \mid \gcd(a, b, c, d)$ which is a contradiction. If $p = 3$ we do not gain any information from the second matrix but from the first we see that $a^2 + b^2 \equiv c^2 + d^2 \equiv 0 \pmod{3}$. Since 0 and 1 are the only squares modulo three it follows that 3 divides $\gcd(a, b, c, d)$, leading to a contradiction also in this case. We conclude that $|r| = 1$ and if $r = 1$ then $M \in \mathrm{PSL}_2(\mathbb{Z})$ and if $r = -1$ then $M \in \mathrm{JPSL}_2(\mathbb{Z})$. \square

Since the maps from forms on the modular group which we are interested in are precisely those from $\mathrm{PGL}_2(\mathbb{Q})$ we immediately deduce the following lemma.

Lemma 3.3. *If $\varphi \in \mathcal{M}(1)$ and $\tilde{\varphi} = \varphi|_A \in \mathcal{M}(\Gamma^3)$ for some $A \in \mathrm{PGL}_2(\mathbb{Q})$ then $\tilde{\varphi} = \varphi$.*

Proposition 3.4. *Let $N_{\Gamma^3}^n(T)$ denote the number of Laplace eigenvalues $\lambda = \frac{1}{4} + R^2$, counted with multiplicity, on $\Gamma^3 \backslash \mathbb{H}$ such that the associated space of Maass waveforms consists of newforms and such that $0 \leq R \leq T$. Then*

$$N_{\Gamma^3}^n(T) \sim \frac{1}{6}T^2 + O(T \ln T) \quad \text{as } T \rightarrow \infty.$$

Proof. Comparing the main terms of Weyl's law (2.8) as $T \rightarrow \infty$ for the modular group, $N_{\mathrm{PSL}_2(\mathbb{Z})}(T) \sim \frac{1}{12}T^2$, with that of Γ^3 , $N_{\Gamma^3}(T) \sim \frac{1}{4}T^2$ (see Venkov [65]), together with Lemma 3.3 we see immediately that the counting function for newforms on Γ^3 satisfies $N_{\Gamma^3}^n(T) = \frac{1}{6}T^2 + O(T \ln T)$ as $T \rightarrow \infty$. \square

3.3. Maps from $\mathcal{M}(\Gamma^3)$ to $\mathcal{M}(\Gamma_0(9))$. From the fact that T acts unitarily on $\mathcal{M}(\Gamma^3)$ it follows immediately that $T^{1/3} = V_3^{-1}TV_3$ is a unitary operator on $\mathcal{M}(9)$ and since T has order 3 on $\mathcal{M}(\Gamma^3)$ we know that $T^{1/3}$ has order three on $\mathcal{M}(\Gamma_0(9))$. Hence we have an orthogonal decomposition of $\mathcal{M}(9)$ analogous to (3.1):

$$(3.2) \quad \mathcal{M}(9, R) = \mathcal{M}(9, R)^{(0)} \oplus \mathcal{M}(9, R)^{(1)} \oplus \mathcal{M}(9, R)^{(-1)}$$

corresponding to the eigenvalue 1, ζ_3 and $\zeta_3^{-1} = \zeta_3^2$ of $T^{1/3}$. Since $V_3TV_3^{-1} = T^{1/3}$ it is clear that the subspace $\mathcal{M}(\Gamma^3, R)^{(m)}$ is mapped into $\mathcal{M}(9, R)^{(m)}$ under the map $\varphi \mapsto \varphi|_{V_3}$. Using the notation $\mathcal{M}^n(9, R)^{(m)} := \mathcal{M}^n(9, R) \cap \mathcal{M}(9, R)^{(m)}$ it follows from Proposition

3.7 and Lemma 3.4 that $\mathcal{M}^n(9, R)^{(m)}$ (for $m = \pm 1$) is non-empty for an infinite number of values of R . Consider now a newform φ on Γ^3 and its associated form $\varphi|_{V_3}$ on $\Gamma_0(9)$ (cf. Example 2.6). We want to show that $\varphi|_{V_3}$ is also a newform on $\Gamma_0(9)$ (Proposition 3.7). To prove this we need to show that $\varphi|_{V_3}$ is orthogonal to all oldforms. Unfortunately, it turns out that the space of oldforms is not invariant under $T^{1/3}$, thus we are not able to simply use the orthogonal decomposition (3.2), but are instead forced to compute the inner products directly.

Lemma 3.5. *If $\varphi \in \mathcal{M}(\Gamma^3)$ then $\psi = \varphi|_{V_3} \in \mathcal{M}(9)$. Furthermore, the map $\varphi \mapsto \varphi|_{V_3}$ commutes with the normalizers T and $T^{1/3}$. That is, if $\varphi|_T = \mu\varphi$ then $\psi|_{T^{1/3}} = \mu\psi$.*

Proof. Since $\Gamma(3) = V_3\Gamma_0(9)V_3^{-1} \subset \Gamma^3$ it follows that $\psi \in \mathcal{M}(9)$ and since $V_3^{-1}TV_3 = T^{1/3}$ we have $\varphi|_{V_3|T^{1/3}} = \varphi|_{T|V_3}$. \square

Lemma 3.6. *If $f \in \mathcal{M}^n(\Gamma^3, R)$ and $g \in \mathcal{M}(3, R)$ then f and $f|_{V_3}$ are orthogonal to g with respect to the Petersson inner product on $\Gamma(3)$ and $\Gamma_0(9)$, respectively.*

Proof. Since f is a newform we can assume, without loss of generality, that $f|_T = \zeta_3 f$. Let \mathcal{F}_0 be a fundamental domain for $\Gamma_0(3)$. By Example 2.2 we see that $\mathcal{F} = \mathcal{F}_0 \sqcup T\mathcal{F}_0 \sqcup T^2\mathcal{F}_0$ is a fundamental domain for $\Gamma(3)$. Since $g|_T = g$ we get

$$\langle f, g \rangle_{\Gamma(3)} = \int_{\mathcal{F}} f \bar{g} d\mu = \int_{\mathcal{F}_0} f \bar{g} d\mu + \int_{T\mathcal{F}_0} f \bar{g} d\mu + \int_{T^2\mathcal{F}_0} f \bar{g} d\mu = \sum_{i=0}^2 \zeta_3^i \cdot \int_{\mathcal{F}_0} f \bar{g} d\mu = 0.$$

By Example 2.1 we know that $\Gamma_0(9) \backslash \Gamma_0(3) \simeq \{\text{Id}, P_2, P_3\}$ with $P_2 = ST^{-3}S$ and $P_3 = P_2^2 = ST^{-6}S$. Since $V_3P_2^{-1}V_3^{-1} = STS$ and $V_3P_3^{-1}V_3^{-1} = ST^2S$ it follows that $f|_{V_3P_2^{-1}} = f|_{STS} = \zeta_3 f|_{V_3}$ and $f|_{V_3P_3^{-1}} = f|_{ST^2SV_3} = \zeta_3^2 f|_{V_3}$. The same argument as above shows that

$$\langle f|_{V_3}, g \rangle_{\Gamma_0(9)} = (1 + \zeta_3 + \zeta_3^2) \int_{\mathcal{F}_0} f|_{V_3} \bar{g} d\mu = 0. \quad \square$$

\square

Proposition 3.7. *If $f \in \mathcal{M}^n(\Gamma^3, R)$ then $f|_{V_3} \in \mathcal{M}^n(9, R)$.*

Proof. The old space $\mathcal{M}^o(9, R)$ is spanned by elements of the form g and $g|_{V_3}$ with $g \in \mathcal{M}(3, R)$. Since $\Gamma(3) = V_3\Gamma_0(9)V_3^{-1}$ we have

$$V_3^{-1}(\Gamma_0(9) \backslash \mathbb{H}) = V_3\Gamma_0(9)V_3^{-1} \backslash \mathbb{H} = \Gamma(3) \backslash \mathbb{H}.$$

Using Lemma 3.6 we now see that $\langle f|_{V_3}, g|_{V_3} \rangle_{\Gamma_0(9)} = \langle f, g \rangle_{\Gamma(3)} = 0$ and $\langle f|_{V_3}, g \rangle_{\Gamma_0(9)} = 0$. \square

Combining Proposition 3.4, Proposition 3.7 and Weyl's law (2.8) for the modular group and $\Gamma_0(3)$ a simple inclusion–exclusion argument shows that the counting function for newforms on $\Gamma_0(9)$ has the main term $\frac{5}{12}T^2$ and that the counting function for twists has main term $\frac{3}{12}T^2$. Hence two fifth of all newforms on $\Gamma_0(9)$ come from Γ^3 . By Lemma 3.1 we know that the multiplicity of the eigenspaces of Maass waveforms on Γ^3 is at least two. Hence we can now prove one of the experimentally motivated conjectures we set out to prove, that is, the following proposition.

Proposition 1.3. *Two fifth of the new part of the spectrum of $\Gamma_0(9)$ has multiplicity at least two.*

3.4. Twists of newforms. In the previous section we showed that newforms on Γ^3 are mapped to newforms on $\Gamma_0(9)$. In this section we will see that twisting of newforms from $\Gamma_0(3)$ and $\mathrm{PSL}_2(\mathbb{Z})$ also produce newforms of $\Gamma_0(9)$. Furthermore, we will see that this contribution is orthogonal to the contribution from Γ^3 .

Lemma 3.8. *If ϕ is a Maass newform on $\mathrm{PSL}_2(\mathbb{Z})$ or $\Gamma_0(3)$ and $\chi = \left(\frac{\cdot}{3}\right)$ then ϕ_χ is a Maass newform on $\Gamma_0(9)$. Furthermore, ϕ_χ is orthogonal to the image of $\mathcal{M}(\Gamma^3, R)$ under V_3 .*

Proof. That $\phi_\chi \in \mathcal{M}^n(9, R)$ follows from the analogue of [3, Thm. 6] for Maass waveforms. The key point of the proof is the multiplicity one theorem for Hecke eigenforms. For the benefit of the reader we include this as Theorem 7.1 below. Consider now $f \in \mathcal{M}^n(\Gamma^3, R)^{(1)}$ and set $F = f|_{V_3}$. By Lemma 3.5 and Proposition 3.7 we know that $F \in \mathcal{M}^n(9, R)^{(1)}$, i.e. that $F|_{T^{1/3}} = \zeta_3 F$ and $\langle F, \phi \rangle_{\Gamma_0(9)} = 0$. Using that $i\sqrt{3}\phi_\chi|_{T^{-1/3}} = \phi - \phi|_{T^{1/3}}$ it follows that

$$\begin{aligned} \langle F, \phi_\chi \rangle_{\Gamma_0(9)} &= \zeta_3^{-1} \langle F|_{T^{1/3}}, \phi_\chi \rangle_{\Gamma_0(9)} = \zeta_3^{-1} \langle F, \phi_\chi|_{T^{-1/3}} \rangle_{\Gamma_0(9)} \\ &= \frac{1}{i\sqrt{3}} \zeta_3^{-1} \langle F, \phi \rangle_{\Gamma_0(9)} - \frac{1}{i\sqrt{3}} \zeta_3^{-1} \langle F, \phi|_{T^{1/3}} \rangle_{\Gamma_0(9)} \\ &= -\frac{1}{i\sqrt{3}} \zeta_3^{-1} \langle F, \phi|_{T^{1/3}} \rangle_{\Gamma_0(9)} = -\frac{1}{i\sqrt{3}} \zeta_3^{-1} \langle F|_{T^{-1/3}}, \phi \rangle_{\Gamma_0(9)} \\ &= -\frac{1}{i\sqrt{3}} \zeta_3^{-2} \langle F, \phi_\chi \rangle_{\Gamma_0(9)} \end{aligned}$$

and since $-\frac{1}{i\sqrt{3}}\zeta_3^{-2} \neq 1$ we conclude that $\langle F, \phi_\chi \rangle_{\Gamma_0(9)} = 0$. □

Using the results of the current section together with the standard newform theory, as introduced in Section 2.4, we can now classify all relevant maps into $\mathcal{M}(\Gamma_0(9))$.

Lemma 3.9. *The following maps are injections into $\mathcal{M}(\Gamma_0(9), R)$:*

$$\begin{aligned} V_3, V_9, \chi_3 : \mathcal{M}(\Gamma_0(1), R) &\rightarrow \mathcal{M}(\Gamma_0(9), R), \\ V_3, \chi_3 : \mathcal{M}^n(\Gamma_0(3), R) &\rightarrow \mathcal{M}(\Gamma_0(9), R), \\ V_3 : \mathcal{M}^n(\Gamma^3, R) &\rightarrow \mathcal{M}(\Gamma_0(9), R). \end{aligned}$$

Furthermore, the images of these maps are pair-wise orthogonal.

4. THE MAIN THEOREM

In section 2.5 we saw that Maass waveforms on Γ^3 maps to forms on $\Gamma_0(9)$ in much the same way as forms on $\Gamma_0(3)$ and $\Gamma_0(1)$ do, i.e. using maps of the form V_d . Due to the standard definition of newforms, we are unfortunately left with the newforms coming from Γ^3 being counted in the space of newforms on $\Gamma_0(9)$. This, together with the fact that also twists of newforms from $\Gamma_0(1)$ and $\Gamma_0(3)$ belong to the space of newforms on $\Gamma_0(9)$ (cf. Lemma 3.8) demonstrates that the traditional definition of the space of newforms needs to be modified in this case.

Consider the group $\Gamma_0(N)$ and recall that oldforms on $\Gamma_0(N)$ are obtained by maps $V_d = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{PGL}_2(\mathbb{Q})$ with $d|N$. For the purpose of extending this definition we first define the space of twists, $\mathcal{M}^t(9, R)$, as the subspace of $\mathcal{M}^n(N, R)$ spanned by elements of the form f_χ with $f \in \mathcal{M}(d, R)$ with $d = 1$ or 3 and χ running through all Dirichlet characters of conductor q such that f_χ belongs to $\mathcal{M}(N, R)$ (cf. Definition 2.7). We then define the space of generalized oldforms, $\mathcal{M}^{\mathrm{go}}(9, R)$, as the space spanned by $\mathcal{M}^t(9, R)$ together with functions of the form $f|_A$ where $f \in \mathcal{M}(\Gamma, R)$ for some subgroup of the modular group Γ and where A runs through elements of $\mathrm{PGL}_2(\mathbb{Q})$ satisfying $A\Gamma_0(N)A^{-1} \subseteq \Gamma$.

We now define the space of *genuinely new* Maass waveforms, $\mathcal{M}^{\mathrm{gn}}(N, R)$, as the orthogonal complement of $\mathcal{M}^{\mathrm{go}}(N, R)$ in $\mathcal{M}(N, R)$. Since this is a subspace of the space of newforms we extend the standard convention and say that f is a genuinely new Maass cusp form if it is also an eigenfunction of all Hecke operators as well as of the reflection J . Setting $\mathcal{M}_{\Gamma^3}(9, R) = V_3 \mathcal{M}(\Gamma^3, R)$ we can now state the Main Theorem precisely.

Theorem 1.1. *The following is an orthogonal decomposition of the space of Maass waveforms on $\Gamma_0(9)$:*

$$\mathcal{M}(9, R) = \mathcal{M}^o(9, R) \oplus \mathcal{M}^t(9, R) \oplus \mathcal{M}_{\Gamma^3}(9, R).$$

Proof. By Lemma 3.9 we can write

$$\mathcal{M}(9, R) = \mathcal{M}^o(9, R) \oplus \mathcal{M}^t(9, R) \oplus \mathcal{M}_{\Gamma^3}(9, R) \oplus \mathcal{M}^{\mathrm{gn}}(9, R)$$

where the last summand, $\mathcal{M}^{\mathrm{gn}}(9, R)$, is empty by Lemma 5.1. □

From this theorem we immediately deduce that there are no genuinely new forms on $\Gamma_0(9)$, i.e. Corollary 1.2.

5. THE SELBERG TRACE FORMULA FOR SUBGROUPS

In this section we will derive the explicit form of the terms in the Selberg trace formula (2.7) for the groups Γ^3 , $\Gamma_0(3)$ and $\Gamma_0(9)$. The reader is encouraged to review the notation of Section 2.6.

Let Γ be a congruence subgroup with $v_2(\Gamma)$ respectively $v_3(\Gamma)$ elliptic conjugacy classes of orders 2 and 3, $\kappa(\Gamma)$ parabolic conjugacy classes and index $\mu(\Gamma) := [\mathrm{PSL}_2(\mathbb{Z}) : \Gamma] < \infty$. It is now straightforward to verify (cf. e.g. [26, pp. 313-314]) that

$$\begin{aligned} I(\Gamma) &= \mu(\Gamma) I_1, \\ E(\Gamma) &= v_2(\Gamma) E_1(2) + v_3(\Gamma) E_1(3) \text{ and} \\ H(\Gamma) &= \sum_n c_n(\Gamma) g(x_n). \end{aligned}$$

It is worth repeating that the sequence $\{x_n\}$ is bounded from below by x_0 – the length of the shortest geodesic on the modular surface, and that the $c_n(\Gamma)$ are non-negative. The parabolic contribution $P(\Gamma)$ is exactly the same as for the modular group (2.10), except that φ_1 (2.11) is replaced by φ_Γ , the determinant of the scattering matrix for Γ . That is:

$$\begin{aligned} P(\Gamma) &= \frac{1}{4} h(0) \left[1 - \varphi_\Gamma \left(\frac{1}{2} \right) \right] - \kappa g(0) \ln 2 + \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) \frac{\varphi'_\Gamma}{\varphi_\Gamma} \left(\frac{1}{2} + ir \right) dr \\ &\quad - \frac{\kappa}{2\pi} \int_{-\infty}^{\infty} h(r) \frac{\Gamma'(1+ir)}{\Gamma(1+ir)} dr. \end{aligned}$$

To relate the parabolic contribution of a subgroup to that of the modular group we need to express φ_Γ in terms of φ_1 . If χ is an even Dirichlet character with conductor q then its associated L-function, $L_\chi(s)$, and completed L-function, $\Lambda_\chi(s)$, are given by

$$L_\chi(s) = \sum_{n=1}^{\infty} \chi(n) n^{-s}, \quad \text{and} \quad \Lambda_\chi(s) = \left(\frac{\pi}{q} \right)^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) L_\chi(s).$$

For the purpose of writing down an explicit formula for the scattering matrix in the manner of Huxley [30] we use a scaled completed L-function

$$\tilde{\Lambda}_\chi(s) = q^{-\frac{s}{2}} \Lambda_\chi(s).$$

Let $\chi = \chi_{0,q}$ denote the principal character modulo q , i.e. $\chi_{0,q}(n) = 1$ if $(n, q) = 1$ and otherwise 0. From the Euler product expansion we see that $L_{\chi_{0,q}}(s) = \zeta(s) \prod_{p|q} (1 - p^{-s})$

and

$$\tilde{\Lambda}_{\chi_{0,q}}(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) L_{\chi_{0,q}}(s) = \prod_{p|q} (1 - p^{-s}) \Lambda(s).$$

To simplify our later formulas we define the quotient $\mathcal{Q}_{\chi}(s) = \frac{\tilde{\Lambda}_{\chi}(2-2s)}{\tilde{\Lambda}_{\chi}(2s)}$ and observe that

$$\mathcal{Q}_{\chi_{0,q}}(s) = \prod_{p|q} \frac{1 - p^{2s-2}}{1 - p^{-2s}} \frac{\Lambda(2-2s)}{\Lambda(2s)} = \prod_{p|q} \frac{1 - p^{2s-2}}{1 - p^{-2s}} \mathcal{Q}_{\chi_{0,1}}(s).$$

For our purposes we need the cases $q = 3$ and 9 :

$$\mathcal{Q}_{\chi_{0,3}}(s) = \frac{1 - 3^{2s-2}}{1 - 3^{-2s}} \mathcal{Q}_{\chi_{0,1}}(s) \quad \text{and} \quad \mathcal{Q}_{\chi_{0,9}}(s) = \frac{1 - 3^{2s-2}}{1 - 3^{-2s}} \mathcal{Q}_{\chi_{0,1}}(s).$$

It is now possible to use a formula for the scattering determinant of $\Gamma^0(N)$ (which is conjugate to $\Gamma_0(N)$) from Huxley [30, p. 147] to show that:

$$\varphi_{\Gamma_0(N)}(s) = A(N)^{1-2s} \prod_{(\chi, m) \in F(N)} \mathcal{Q}_{\chi^2 \chi_{0,m}}(s)$$

where

$$A(N) = \prod_{(\chi, m) \in F} \frac{q_{\chi} N}{(m, N/m)}$$

and the product is taken over the set

$$F(N) = \{(\chi, m) \mid m|N, q_{\chi}|m, q_{\chi}m|N \text{ and } \chi \text{ is a primitive Dirichlet character mod } q_{\chi}\}.$$

(The difference between our and Huxley's constant $A(N)$ arises from a difference in the normalization of the completed Dirichlet L-functions). For $N = 3$ and 9 we have

$$F(3) = \{(\chi_{0,1}, 1), (\chi_{0,1}, 3)\}, A(3) = 9,$$

$$F(9) = \{(\chi_{0,1}, 1), (\chi_{0,1}, 3), (\chi_3, 3), (\chi_{0,1}, 9)\} \text{ and } A(9) = 3^7.$$

For all $(\chi, m) \in F(3) \cup F(9)$ we see that $\chi^2 = \chi_{0,q}$ and since $q|m$ we get $\chi^2 \chi_{0,m} = \chi_{0,m}$ for $m = 1, 3$ and $\chi^2 \chi_{0,9} = \chi_{0,3}$. Using this in the formula above we get

$$\varphi_{\Gamma_0(3)}(s) = 9^{1-2s} \mathcal{Q}_{\chi_{0,1}}(s) \mathcal{Q}_{\chi_{0,3}}(s) = 9^{1-2s} \frac{1 - 3^{2s-2}}{1 - 3^{-2s}} \varphi_{\Gamma_0(1)}^2(s) \quad \text{and}$$

$$\varphi_{\Gamma_0(9)}(s) = (3^7)^{1-2s} \mathcal{Q}_{\chi_{0,1}}(s) \mathcal{Q}_{\chi_{0,3}}(s)^3 = (3^7)^{1-2s} \left(\frac{1 - 3^{2s-2}}{1 - 3^{-2s}} \right)^3 \varphi_{\Gamma_0(1)}^4(s).$$

The final component we need is the scattering determinant for Γ^3 . By Venkov [65] we have

$$\varphi_{\Gamma^3}(s) = 3^{1-2s} \varphi_{\Gamma_0(1)}(s).$$

We can now compute the parabolic contribution for the various groups. For simplicity we introduce the following notation:

$$J_0 = \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) dr \quad \text{and}$$

$$J_3 = \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) \frac{d}{ds} \ln \left(\frac{1-3^{2s-2}}{1-3^{-2s}} \right) \Big|_{s=\frac{1}{2}+ir} dr.$$

Then, using the fact that $\varphi_{\Gamma_0(3)}\left(\frac{1}{2}\right) = \varphi_{\Gamma_0(9)}\left(\frac{1}{2}\right) = \varphi_{\Gamma^3}\left(\frac{1}{2}\right) = \varphi_1\left(\frac{1}{2}\right) = 1$, we have

$$P_1 = -g(0) \ln 2 + \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) \frac{\varphi'_{\Gamma_0(1)}\left(\frac{1}{2}+ir\right)}{\varphi_{\Gamma_0(1)}\left(\frac{1}{2}+ir\right)} dr - \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) \frac{\Gamma'(1+ir)}{\Gamma(1+ir)} dr,$$

$$P(\Gamma^3) = P_1 - 2 \ln 3 J_0,$$

$$P(3) = 2P_1 - 4 \ln 3 J_0 + J_3 \quad \text{and}$$

$$P(9) = 4P_1 - 14 \ln 3 J_0 + 3J_3.$$

5.1. Comparison of terms for the genuinely new trace formula. To obtain a trace formula for the genuinely new part of the spectrum we need to compare the full contribution of $\Gamma_0(9)$ with the contribution of the images of the maps in Lemma 3.9. Let $X(\Gamma)$, $X^n(\Gamma)$ and $X^{\text{gn}}(\Gamma)$ denote a term in the trace formula ($X = I, E, P$ or H) and the corresponding contribution from the newforms and genuinely new forms. Also let $X^*(d) = X^*(\Gamma_0(d))$. The contribution of $\text{PSL}_2(\mathbb{Z})$ to $X(9)$ is then 3 times for the old forms and once for the twists and similarly, the new forms on $\Gamma_0(3)$ contributes 3 times and the new forms on Γ^3 twice. With this notation the genuinely new term is therefore given by

$$\begin{aligned} X^{\text{gn}}(9) &= X(9) - 4X_1 - 3X^n(3) - X^n(\Gamma^3) \\ &= X(9) - 4X_1 - 3[X(3) - 2X_1] - [X(\Gamma^3) - X_1] \\ &= X(9) + 3X_1 - 3X(3) - X(\Gamma^3). \end{aligned}$$

For the groups Γ^3 , $\Gamma_0(3)$ and $\Gamma_0(9)$ the tuple of data (μ, κ, v_2, v_3) is given by $(3, 1, 3, 0)$, $(4, 2, 0, 1)$ and $(12, 4, 0, 0)$, respectively. Hence

$$\begin{aligned} I^{\text{gn}}(9) &= I_1 [\mu(9) + 3 - 3\mu(3) - \mu(\Gamma^3)] = 0 \quad \text{and} \\ E^{\text{gn}}(9) &= 0 + 3E_1(2) + 3E_1(3) - 3E_1(3) - 3E_1(2) = 0, \\ P^{\text{gn}}(9) &= 4P_1 - 14 \ln 3 J_0 + 3J_3 + 3P_1 - 3(2P_1 - 4 \ln 3 J_0 + J_3) - P_1 + 2 \ln 3 J_0 = 0. \end{aligned}$$

Thus all terms except for possibly the hyperbolic ones cancel completely. Let $\sigma_{\text{gn}} \subset \sigma(\Gamma_0(9))$ denote the genuinely new part of the spectrum of $\Gamma_0(9)$. The trace formula

(2.7) reduces to the following relation between the genuinely new spectrum and a sum over lengths of geodesics:

$$(5.1) \quad \sum_{r_n \in \sigma_{\text{gn}}} h(r_n) = \sum_n c_n g(x_n)$$

where $\{c_n\}$ and $\{x_n\}$ are sequences in \mathbb{R} with $x_n \geq x_0 > 0$ (cf. Section 2.6). To show that this part also vanishes we will make a specific choice of test function. Let $T > 0$ and consider

$$h_T(r) = \left(\frac{\sin Tr}{Tr} \right)^4.$$

It is readily verified that h_T is even and analytic and that $h_T(r) = O(|r|^{-4})$ as $|\Re(r)| \rightarrow \infty$ in any strip of the form $|\Im(r)| < \delta$. Let g_T be the Fourier transform of h_T . Then g_T is easily found to be a convolution of triangle functions and it has support contained in $[-\frac{2}{T}, \frac{2}{T}]$. Let $T > T_0 = \frac{2}{x_0}$ and consider the trace formula (5.1) above. Then

$$\sum_{r_n \in \sigma_{\text{gn}}} h_T(r_n) = \sum_n c_n g_T(x_n)$$

and the right hand side is zero since $x_n > \frac{2}{T}$ and thus the right hand side is also zero. Since h_T is non-negative and this holds for all $T > T_0$ it follows that the set σ_{gn} is empty. We have thus shown the following lemma.

Lemma 5.1. *The space $\mathcal{M}^{\text{gn}}(9, R)$ is empty for all R .*

6. HECKE OPERATORS AND NEWFORMS ON $\Gamma_0(9)$

We now want to study how the decomposition (3.2), into eigenspaces of the operator $T^{1/3}$, behaves under the action of the Hecke operators T_p defined in Section 2.4. It turns out that the mapping properties of T_p depend on p modulo 3 and we have the following lemma.

Lemma 6.1. *The family of Hecke operators $\{T_p\}_{p \neq 3}$ on $\Gamma_0(9)$ has the following properties:*

- (a) $JT_pJ = T_p$ for all primes $p \neq 3$,
- (b) $T_p T^{1/3} = T^{1/3} T_p$ if $p \equiv 1 \pmod{3}$ and
- (c) $T_p T^{1/3} = T^{-1/3} T_p$ if $p \equiv 2 \pmod{3}$.

Proof. Using Definition 2.4 we can write the action of T_p on $f \in \mathcal{M}(9)$ as

$$T_p f = \sum_{b \pmod{p}} f|_{\beta_{p,b}} + f|_{\alpha_p}$$

where $\alpha_p = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ and $\beta_{p,b} = \begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix}$. A direct computation shows that $J\beta_{p,j}J = \beta_{p,-j}$ and $J\alpha_pJ = \alpha_p$. It follows that T_p commutes with J for all primes p . By representing the operator $T^{1/3}$ by the matrix $\begin{pmatrix} 1 & \frac{1}{3} \\ 0 & 1 \end{pmatrix}$ we check that if $p \equiv 1 \pmod{3}$ then $\beta_{p,b}T^{\frac{1}{3}} = T^{\frac{1}{3}}\beta_{p,b'}$ with $b' \equiv b + \frac{1-p}{3} \pmod{p}$ and $\alpha_pT^{\frac{1}{3}} = T^{\frac{p-1}{3}}T^{\frac{1}{3}}\alpha_p$. If $p \equiv 2 \pmod{3}$ then $\beta_{p,b}T^{\frac{1}{3}} = T^{\frac{2}{3}}\beta_{p,b'}$ with $b' \equiv b + \frac{2-p}{3} \pmod{p}$ and $\alpha_pT^{\frac{1}{3}} = T^{\frac{p-2}{3}}T^{\frac{2}{3}}\alpha_p$. \square

From this lemma we conclude that if $p \equiv 1 \pmod{3}$ then

$$T_p : \mathcal{M}^n(9, R)^{(m)} \rightarrow \mathcal{M}^n(9, R)^{(m)} \quad \text{for } m = 0, 1, -1$$

and if $p \equiv 2 \pmod{3}$ then

$$\begin{aligned} T_p : \mathcal{M}^n(9, R)^{(0)} &\rightarrow \mathcal{M}^n(9, R)^{(0)} \quad \text{and} \\ T_p : \mathcal{M}^n(9, R)^{(\pm 1)} &\rightarrow \mathcal{M}^n(9, R)^{(\mp 1)}. \end{aligned}$$

Remark 6.2. We disregard the prime $p = 3$ in this discussion because T_3 is identically zero on $\mathcal{M}^n(\Gamma_0(9), R)^{(\pm 1)}$. To verify this, observe that $T^{\frac{1}{3}}\beta_{3,j} = \beta_{3,j+1}$ and hence $T_3T^{1/3} = T_3$. Therefore, either $T^{1/3}$ acts trivially or T_3 acts as the zero operator.

Since the decomposition into eigenspaces of $T^{1/3}$ is a key step in proving multiplicity we consider a modified family of Hecke operators which preserves this decomposition:

$$\mathcal{T}_p = T_p \quad \text{if } p \equiv 1 \pmod{3} \quad \text{and} \quad \mathcal{T}_p = JT_p \quad \text{if } p \equiv 2 \pmod{3}.$$

Lemma 6.3. *The family of operators $\mathcal{T} = \{\mathcal{T}_p\}_{p \neq 3}$ has the following properties:*

- (1) All \mathcal{T}_p are normal and pair-wise commuting.
- (2) All \mathcal{T}_p preserves $\mathcal{M}^n(9, R)^{(m)}$ for $m = 0, 1, -1$.

Proof. The first property follows from the corresponding property of the Hecke operators T_p (cf. e.g. [39, ch. 4.5]) together with the observations that $J^2 = \text{Id}$ and $p^2 \equiv 1 \pmod{3}$ for any prime $p \neq 3$. The second property follows from Lemma 6.1 combined with the fact that J intertwines the spaces with $m = 1$ and -1 . \square

Since \mathcal{T} consists of commuting normal (recall that an operator on a Hilbert space is said to be normal if it commutes with its adjoint) operators it follows that $\mathcal{M}^n(9, R)^{(1)}$ has an orthonormal basis $\{\Phi_1, \dots, \Phi_h\}$ of simultaneous eigenfunctions of all operators in \mathcal{T} . If $\Psi_k = \Phi_{k|J}$ then $\{\Psi_1, \dots, \Psi_h\}$ is also an orthonormal \mathcal{T} -eigenbasis of $\mathcal{M}^n(9, R)^{(-1)}$. We now define

$$F_j^+ = \frac{1}{2}(\Phi_j + \Psi_j), \quad \text{and} \quad F_j^- = \frac{1}{2}(\Phi_j - \Psi_j), \quad 1 \leq j \leq h.$$

It is easy to verify that $\{F_j^\pm\}_{1 \leq j \leq h}$ is an orthonormal \mathcal{T} -eigenbasis of

$$\bigoplus_{m=\pm 1} \mathcal{M}^n(9, R)^{(m)}$$

and that $JF_j^\pm = \pm F_j^\pm$. Hence F_j^+ and F_j^- are eigenforms of the Hecke operators T_p , $p \neq 3$ and hence also of all T_n with $(n, 3) = 1$. Suppose that Φ_j has a Fourier expansion

$$\Phi_j(z) = \sum_{n \neq 0} a_j(n) \kappa_n(y) e(nx)$$

where $\kappa_n(y) = \sqrt{y} K_{iR}(2\pi|n|y)$ and $e(x) = e^{2\pi i x}$. Then

$$\begin{aligned} \Phi_{j|T^{1/3}}(z) &= \sum_{n \neq 0} a_j(n) \kappa_n(y) e(n(x + \tfrac{1}{3})) = \sum_{n \neq 0} a_j(n) \zeta_3^n \kappa_n(y) e(nx) \quad \text{and} \\ \Psi_j(z) &= \Phi_j(-x + iy) = \sum_{n \neq 0} a_j(n) \kappa_n(y) e(-nx) = \sum_{n \neq 0} a_j(-n) \kappa_n(y) e(nx). \end{aligned}$$

Since $\Phi_{j|T^{1/3}} = \zeta_3 \Phi$ it follows that $a_j(n) = 0$ unless $n \equiv 1 \pmod{3}$ and

$$\begin{aligned} F_j^+(z) &= \sum_{(n,3)=1} c_j^+(n) \kappa_n(y) e(nx), \quad c_j^+(n) = \begin{cases} a_j(n), & n \equiv 1 \pmod{3}, \\ a_j(-n), & n \equiv 2 \pmod{3}, \end{cases} \\ F_j^-(z) &= \sum_{(n,3)=1} c_j^-(n) \kappa_n(y) e(nx), \quad c_j^-(n) = \begin{cases} a_j(n), & n \equiv 1 \pmod{3}, \\ -a_j(-n), & n \equiv 2 \pmod{3}. \end{cases} \end{aligned}$$

In other words, we have $c_j^-(n) = \left(\frac{n}{3}\right) c_j^+(n)$ for all n . The functions F_j^\pm are not identically zero since Φ_j is not identically zero. Furthermore, F_j^+ is orthogonal to F_j^- since J is an involution. Proposition 1.4 of the introduction (repeated below) is now an immediate consequence of the construction of the basis $\{F_j^\pm\}$ of $\bigoplus_{m=\pm 1} \mathcal{M}^n(9, R)^{(m)}$ together with Weyl's law for newforms on Γ^3 , cf. Lemma 3.4.

Proposition 1.4. *There exist an infinite number of pairs $\{F^+, F^-\}$ of Maass newforms on $\Gamma_0(9)$ with the property that the Hecke eigenvalues $c^+(n)$ and $c^-(n)$ of F^+ and F^- are related through*

$$c^-(n) = \left(\frac{n}{3}\right) c^+(n)$$

for all n relatively prime to 3.

7. SOME PERSPECTIVES FROM REPRESENTATION THEORY

7.1. A representation-theoretical interpretation of the main theorem. It is a well-known fact that a modular form f of weight k on a subgroup $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{Z})$ corresponds

to a vector-valued modular form F on $\mathrm{PSL}_2(\mathbb{Z})$, transforming with respect to a certain finite dimensional representation ρ_F . To be more precise, if $\Gamma \backslash \mathrm{PSL}_2(\mathbb{Z}) = \{V_i\}$ and F is taken as the vector with components $F_i = f|_k V_i$ then F will transform according to the induced representation of Γ . The matrix coefficients of this can be defined by $\rho_F(A)_{ij} = 1$ if $V_i A V_j^{-1} \in \Gamma$ and 0 otherwise. The representation ρ_F is in general not irreducible. One way of obtaining information about the original modular form f is to study the decomposition of ρ_F into irreducible components. It turns out that in the present case, that of Maass waveforms on $\Gamma_0(9)$, the representation ρ_F can be viewed as a representation of the symmetric group S_4 . The purpose of this section is to demonstrate how the irreducible representations of S_4 are related to the different types of functions appearing in the decomposition of the space $\mathcal{M}(9)$ given in Theorem 1.1.

For technical reasons it is easier to work with the general linear group rather than the special linear group. If $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{Z})$ then we let $\bar{\Gamma} \subseteq \mathrm{PGL}_2(\mathbb{Z})$ denote the group generated by the elements of Γ together with the reflection J . There are two canonical characters on $\mathrm{PGL}_2(\mathbb{Z})$, namely the trivial character, χ_0 , and the sign of the determinant, $\chi_{\mathrm{sgn}} = \mathrm{sgn} \circ \det$.

Fix a non-zero $f \in \mathcal{M}(9)$ which transforms under the character χ_0 or χ_{sgn} on $\bar{\Gamma}_0(9)$, that is, with the notation used in the introduction, f is either *even* or *odd*. Since $V_3 \Gamma_0(9) V_3^{-1} = \Gamma(3)$ we see that $f|_{V_3^{-1}} \in \mathcal{M}(\Gamma(3))$. Set $v_\gamma = f|_{V_3^{-1} \gamma}$ and consider the complex vector space

$$\mathcal{V}(f) = \{v_\gamma \mid \gamma \in \Gamma(3) \backslash \mathrm{PGL}_2(\mathbb{Z})\}$$

equipped with the following $\mathrm{PGL}_2(\mathbb{Z})$ -action: $A.v_\gamma := v_{\gamma A^{-1}}$. Since $\Gamma(3)$ is a normal subgroup it is clear that the elements γ can be taken as simultaneous right- and left-coset representatives and hence $B.v_\gamma = v_{\gamma B^{-1}} = v_\gamma$ if $B \in \Gamma(3)$. It follows that $\mathcal{V}(f)$ is a complex, non-zero, finite-dimensional representation of $\mathrm{PGL}_2(\mathbb{Z})/\Gamma(3)$.

Let \mathbb{F}_3 denote the finite field with 3 elements and consider the group homomorphism $\varphi_3 : \mathrm{PGL}_2(\mathbb{Z}) \rightarrow \mathrm{PGL}_2(\mathbb{F}_3)$ given by reducing each matrix entry mod 3. Since φ_3 is surjective and its kernel is $\Gamma(3)$ it follows immediately that $\mathrm{PGL}_2(\mathbb{Z})/\Gamma(3) \simeq \mathrm{PGL}_2(\mathbb{F}_3)$. It is easy to see that the maps $\bar{S} := \varphi_3(S)$, $\bar{T} := \varphi_3(T)$ and $\bar{J} := \varphi_3(J)$ are generators of $\mathrm{PGL}_2(\mathbb{F}_3)$. A short calculation shows that the map $h : \mathrm{PGL}_2(\mathbb{F}_3) \rightarrow S_4$ defined on the generators by $h(\bar{S}) = (12)(34)$, $h(\bar{T}) = (123)$ and $h(\bar{J}) = (12)$ gives an isomorphism between $\mathrm{PGL}_2(\mathbb{F}_3)$ and the symmetric group S_4 . Due to this isomorphism we are able to obtain all necessary information about the irreducible representations of $\mathrm{PGL}_2(\mathbb{F}_3)$ from standard references on representations of finite groups, for example Fulton and Harris [21].

It is clear that $\mathcal{V}(f)$ can be viewed as a representation of S_4 , and we know that S_4 has five irreducible representations: χ_0 , χ_{sgn} , WS , ρ_{std} , and $\rho_{\mathrm{std}} \otimes \chi_{\mathrm{sgn}}$, of dimensions 1, 1, 2, 3 and 3, respectively. It remains to find out the exact correspondence between these

irreducible representations and properties of the function f . If f is an oldform related to a specific newform g of lower level then we view the representation generated by *all* oldforms obtained from g as the natural object associated to f , instead of just the representation $\mathcal{V}(f)$ defined above. To be precise, if g has level 1 and $f \in \text{span}\{g, g|_{V_3}, g|_{V_9}\}$ then we set $\mathcal{V}(f) := \mathcal{V}(g, g|_{V_3}, g|_{V_9}) = \{v_{i\gamma} | \gamma \in \Gamma(3) \setminus \text{PGL}_2(\mathbb{Z}), i = 0, 1, 2\}$, where $v_i = g|_{V_3^{i-1}}$, and analogously if g has level 3. By identifying S_4 and $\text{PGL}_2(\mathbb{F}_3)$ as above, and using the character table of S_4 , it is not hard to show the following properties explicitly:

- (1) If f is an oldform associated to an even newform of level 1 then $\mathcal{V}(f) = \chi_0 \oplus \rho_{\text{std}}$.
- (2) If f is an oldform associated to an even newform of level 3 then $\mathcal{V}(f) = \rho_{\text{std}}$.
- (3) If f is the twist of an even newform of level 1 or 3 then $\mathcal{V}(f) = \rho_{\text{std}}$.
- (4) If f is the lift of a newform on Γ^3 then $\mathcal{V}(f) = W$.

In the first three cases, an odd instead of an even newform corresponds to a twist of the representation by χ_{sgn} . To prove 4 it is helpful to use the description of Γ^3 in terms of congruences (2.4) to show that the kernel of W is precisely Γ^3 (viewed as a subgroup of $\text{PGL}_2(\mathbb{F}_3)$). From 1-4 we see that there is a relationship between the irreducible representations of S_4 and the different constituents in the decomposition of $\mathcal{M}(9)$ given by Theorem 1.1. However, to *prove* that such a decomposition holds we would need a one-to-one correspondence, something which seems to be out of reach using the “classical” approach outlined above.

To use representation theory to prove that Theorem 1.1 holds, or equivalently, that there are no genuinely new forms on $\Gamma_0(9)$, we have to consider automorphic representations of $\text{GL}_2(\mathbb{A})$ where \mathbb{A} is the ring of adeles over \mathbb{Q} . A precise formulation of all definitions and results in this area would be too lengthy and take us far out of the scope of this paper. We therefore simply outline the essential parts of the argument and leave the technicalities to the interested reader. The necessary background can be obtained from, for example, Gelbart [22]. Let \mathbb{Z}_p be the ring of integers in \mathbb{Q}_p , the p -adic completion of \mathbb{Q} at the finite prime p . For a positive integer N we choose subgroups $K_0(N)$, $K_0^0(N)$, $K(N)$ and K^3 of $\text{GL}_2(\hat{\mathbb{Z}}) = \prod_{p < \infty} \text{GL}_2(\mathbb{Z}_p)$ satisfying $\text{GL}_2^+(\mathbb{R})K \cap \text{GL}_2(\mathbb{Q}) = \Gamma_0(N)$, $\Gamma_0^0(N)$, $\Gamma(N)$ and Γ^3 , respectively. The choice is made such that the determinant map is surjective to \mathbb{Z}_p^\times , and hence such that strong approximation holds, that is, $\text{GL}_2(\mathbb{A}) = \text{GL}_2(\mathbb{Q})K\text{GL}_2^+(\mathbb{R})$ when K is any of the above groups.

Let π be an automorphic representation and write π^K for the set of vectors of π which are fixed under some $K \subseteq \text{GL}_2(\hat{\mathbb{Z}})$. It is well-known that there is a unique newform attached to each irreducible component of π . If $\pi^{K_0(9)} \neq \{0\}$ then we use $V_3 \in \text{GL}_2^+(\mathbb{R})$ to show that $\pi^{K(3)} \neq \{0\}$ and it is easy to see that $\mathcal{V}(\pi) := \pi^{K(3)}$ can be viewed as a complex, non-zero, finite-dimensional representation of $\text{PGL}_2(\mathbb{F}_3)$. We want to show that none of the possible irreducible constituents of $\mathcal{V}(\pi)$ corresponds to a genuine newform.

It clear that if $\mathcal{V}(\pi)$ contains the trivial representation then it contains a vector fixed under $\mathrm{GL}_2(\hat{\mathbb{Z}})$ and the associated newform has level 1. If $\mathcal{V}(\pi)$ contains ρ_{std} then it is easy to check that it contains a vector which fixed under T and therefore under the whole group $K_0(3)$. Hence the associated newform has level 3. If $\mathcal{V}(\pi)$ contains χ_{sgn} or $\chi_{\mathrm{sgn}} \otimes \rho_{\mathrm{std}}$ then $\mathcal{V}(\pi) \otimes \chi_{\mathrm{sgn}} = \mathcal{V}(\pi \otimes \chi_3)$ contains χ_0 or ρ_{std} , and hence π is a twist of an automorphic representation associated to a newform of level 1 or 3. For the last statement, observe that for $A \in \mathrm{GL}_2(\mathbb{F}_3)$ we have $\chi_3(\det(A)) = \mathrm{sgn}(\det(A))$, and by definition $\pi \otimes \chi_3$ consists of functions of the form $g \mapsto \varphi(g)\chi_3(\det(g))$ with $\varphi \in \pi$. Finally, if $\mathcal{V}(\pi)$ contains W then π contains a non-zero vector fixed under the subgroup $K^3 \subset \mathrm{GL}_2(\hat{\mathbb{Z}})$ and the newform attached to π is therefore a newform on Γ^3 .

Since one of these five possibilities must occur we conclude that any newform attached to π is in fact a newform on some subgroup of lower level, The corresponding “classical” statement for a Maass form, or modular form, f , follows by considering π_f , the representation generated by all the $\mathrm{GL}_2(\mathbb{A})$ -translates of the automorphic form φ_f associated to f . See for example [22, section 5.C].

The connection with representation theory of S_4 was first mentioned to the author by A. Mellit. A more precise description of the idea behind this correspondence, together with details regarding the automorphic point of view was given to the author by K. Buzzard (private communication).

7.2. Multiplicity one. A fundamental property in the theory of holomorphic newforms is that they are uniquely determined by their Hecke eigenvalues, λ_n . In fact, Atkin and Lehner [3, Theorem 4] showed that they are uniquely determined by any infinite subset of the Hecke eigenvalues with prime index. This is the first example of a *multiplicity one* theorem. There has been many generalizations of this theorem, both formulated classically and in in terms of automorphic representations. Using Gelbart [22, Thm. 5.19] to translate results from the language of automorphic representations to that of Maass waveforms we will now describe some of the strongest multiplicity one theorems which are known at the moment.

Theorem 7.1 (Jacquet-Shalika, Gelbart, Ramakrishnan and Rajan). *Let $\varphi \in \mathcal{M}^n(N, R)$ and $\psi \in \mathcal{M}^n(N', R)$ be two Maass newforms with identical Laplace eigenvalue but possibly different levels N and N' . Let λ_p and λ'_p denote their respective T_p -eigenvalues and define S to be the set of primes where these eigenvalues are different:*

$$S = \{p \text{ prime} \mid \lambda_p \neq \lambda'_p\}.$$

Let $D(S)$ be the Dirichlet (or analytic) density of S , defined by

$$D(S) = \lim_{s \rightarrow 1^+} \frac{1}{\ln\left(\frac{1}{s-1}\right)} \sum_{p \in S} p^{-s}.$$

Then

- (a) If S is finite then $N = N'$ and $f = g$.
- (b) If $D(S) < \frac{1}{8}$ then $N = N'$ and $f = g$.
- (c) If $\sum_{p \in S} p^{-\frac{2}{5}} < \infty$ then $N = N'$ and $f = g$.

The „standard“ strong multiplicity one theorem (a) follows from e.g. Jacquet-Shalika [33, Thm. 4.8] or Gelbart [22, Thm. 5.12]. The refinements obtained in (b) and (c) were shown by Ramakrishnan [48] and Rajan [46, pp. 188-189] respectively. See also [45]. A slightly different flavor of multiplicity one is given by the following theorem of Ramakrishnan [49, Cor. 4.1.3]. All notations are as in Theorem 7.1.

Remark 7.2. The above theorem is also true for forms with Nebentypus (Dirichlet characters). In this case, if φ and ψ are assumed to have Nebentypus χ and χ' then the conclusions (a)-(c) also include the statement that $\chi = \chi'$.

Theorem 7.3 (Ramakrishnan). *If $\lambda_p^2 = \lambda_p'^2$ for every prime p , $(NN', p) = 1$ then there exists a Dirichlet character χ such that $\lambda_p' = \chi(p)\lambda_p$.*

From the corresponding results in the l -adic context [44] it is conjectured (cf. e.g. [46, p. 189]) that the “critical density” $\frac{1}{8}$ in Theorem 7.1 (b) above can be replaced with $\frac{1}{2}$ if f and g are not of CM-type.

With notation as in the previous theorems assume that $\lambda_p' = \chi(p)\lambda_p$ with a non-trivial Dirichlet character $\chi \bmod M$. Then $S = S_\chi$ where

$$S_\chi = \{p \text{ prime} \mid \chi(p) \neq 1\} \quad \text{and} \\ D(S_\chi) = \frac{1}{\phi(M)} \left| \{p \in (\mathbb{Z}/M\mathbb{Z})^* \mid \chi(p) \neq 1\} \right|$$

by Serre [54, Thm. 2, Ch. VI]. Since $\sum_{d \bmod M} \chi(d) = 0$ unless χ is trivial we see that the set $\{\chi(d) \mid d \in (\mathbb{Z}/M\mathbb{Z})^*\}$ contains at most $\frac{\phi(M)}{2}$ ones and thus $D(S_\chi) \geq \frac{1}{2}$. It is therefore not expected that the inequality, $D(S) < \frac{1}{8}$, of Theorem 7.1 can be replaced by any inequality stronger than $D(S) < \frac{1}{2}$ even for generic forms.

Now, let $\{F^+, F^-\}$ be a pair of newforms on $\Gamma_0(9)$ as in Proposition 1.4, i.e. having Hecke eigenvalues $\lambda_p^- = \chi_3(p)\lambda_p^+$ with $\chi_3(p) = \left(\frac{p}{3}\right)$. Let π^+ and π^- be the automorphic representations corresponding to F^+ and F^- and let Π^+ and Π^- be the base-change of π^+ and π^- with respect to the quadratic extension $\mathbb{Q}(\zeta_3)$. The associated character of this extension is χ_3 and since $\pi^- = \chi_3 \otimes \pi^+$ it follows that $\Pi^+ = \pi^+ \oplus \chi_3 \otimes \pi^+$ and $\Pi^- = \pi^- \oplus \chi_3 \pi^-$ are equal. Or, in other words, the base-changed forms E^+ and E^- of F^+ and F^- have identical L-functions.

8. CONCLUDING REMARKS AND POSSIBLE EXTENSIONS

A natural question to ask at this point is whether the results of this paper, in particular the non-existence of genuinely new forms and the existence of multiple “new” eigenvalues, are specific to $\Gamma_0(9)$ or if similar arguments are applicable to other (square) levels.

An essential ingredient in the proof of the existence of multiple eigenvalues in the new spectrum on $\Gamma_0(9)$ was the normalizer $T^{1/3}$. In particular, that we could prove the existence of eigenfunctions of this operator, with eigenvalue different from 1. According to Atkin and Lehner [3, Lemma 29] the normalizer of $\Gamma_0(N)$ contains maps of the form $T^{\frac{1}{q}}$ if $q^2 \mid N$ and $q = 2, 3, 4, 8$. Recall the argument which forces a higher multiplicity: if f is an eigenfunction of $T^{1/3}$ with eigenvalue $\mu \neq 1$ then $f|_J$ is also an eigenfunction of $T^{1/3}$, but with eigenvalue $\mu^{-1} \neq \mu$. To prove multiplicity in the case of even q we must prove the existence of eigenfunctions with eigenvalues not equal to 1 or -1 .

For the non-existence of genuinely new Maass forms we have displayed two different approaches, one “classical”, using orthogonality relations and the Selberg trace formula, and one “automorphic”, using representation theory. For both of these approaches an important step was that we found the group Γ^3 between $\mathrm{PSL}_2(\mathbb{Z})$ and $\Gamma_0^3(3)$, with the latter group conjugate to $\Gamma_0(9)$. Because of the simple form of Γ^3 we could prove that newforms on Γ^3 lift to newforms on $\Gamma_0(9)$ and we could also express all relevant terms in the Selberg trace formula explicitly.

To use the classical approach in this paper to study the group $\Gamma_0(N^2)$ for an arbitrary integer N a necessary first step is to determine all its conjugates in $\mathrm{PSL}_2(\mathbb{Z})$, as well as their supergroups and possible lifting maps. Once this is done, a hint to whether there are genuinely new forms or not can be obtained from the indices of the groups involved, together with careful book-keeping. A complete proof clearly requires more effort, and if the intermediate groups are not cycloidal, or otherwise admit simple descriptions, it might be hard to prove all required lemmas. Unfortunately we know by a result of Petersson [41] that there is in fact only a finite number of cycloidal congruence groups, all with indices dividing $55440 = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$. For general N it is therefore always necessary to investigate more complicated groups.

In the representation-theoretical approach described in Section 7.1 the key components are the isomorphisms $\Gamma_0^0(3) \backslash \mathrm{PGL}_2(\mathbb{Z}) \simeq \mathrm{PGL}_2(\mathbb{F}_3)$ and $\mathrm{PGL}_2(\mathbb{F}_3) \simeq S_4$. From these we immediately got an explicit description of all irreducible representations of $\Gamma_0^0(3) \backslash \mathrm{PGL}_2(\mathbb{Z})$. For a prime $p > 3$ the group $\Gamma_0^0(p)$ is no longer equal to $\Gamma(p)$ and we first need to identify $G_p := \Gamma_0^0(p) \backslash \mathrm{PGL}_2(\mathbb{Z})$ as a subgroup of $\mathrm{PGL}_2(\mathbb{F}_p)$. Then we have to find a subgroup G , of some symmetric group S_n , which is isomorphic to G_p . The irreducible representations of G_p (for a given, fixed p) can then be obtained by, for example, a computer algebra

program. The biggest problem is clearly to identify fixed vectors under these irreducible constituents with subgroups of $\mathrm{PSL}_2(\mathbb{Z})$ and possible twists.

In order to give a flavor of how the general case might be treated we briefly discuss the cases $\Gamma_0(p^2)$ with $p = 2$, $p = 5$ and $p = 7$. It is known that there is a group Γ^2 , of level and index 2 (cf. [50, 1.5]), which does not contain $\Gamma_0(4)$. The representation-theoretical approach is very simple in this case since $\Gamma_0(4)$ is conjugate to $\Gamma(2)$ and $\Gamma(2) \backslash \mathrm{PGL}_2(\mathbb{Z}) \simeq \mathrm{PGL}_2(\mathbb{F}_2)$. Furthermore, $\mathrm{PGL}_2(\mathbb{F}_2)$ is isomorphic to the symmetric group S_3 , which only has three irreducible representations: χ_0 , χ_{sgn} and ρ_{std} . It is then easy to show that χ_0 , χ_{sgn} and ρ_{std} corresponds to $\mathrm{PSL}_2(\mathbb{Z})$, Γ^2 , $\Gamma_0(2)$, in the same way that the representations χ_0 , W and ρ_{std} of S_4 corresponded to the subgroups $\mathrm{PSL}_2(\mathbb{Z})$, Γ^3 and $\Gamma_0(3)$ in Section 7.1. Using either this approach, or lemmas analogous to those in Sections 3.2 and 3.3 together with the Selberg trace formula, it is easy to prove the following theorem.

Theorem 8.1. *There are no genuinely new forms on $\Gamma_0(4)$.*

Motivated by a brief numerical investigation into the case of $\Gamma_0(25)$ we conjecture that there are no genuinely new Maass forms on $\Gamma_0(25)$. In the decomposition of $\mathcal{M}(25)$ there are clearly contributions from the oldforms and twists from $\Gamma_0(1)$ and $\Gamma_0(5)$ and lifts from the cycloidal group Γ^5 of index end level 5. There is also an additional contribution from the Maass forms on $\Gamma_0(5)$ with Nebentypus $\chi_5 = \left(\frac{\cdot}{5}\right)$, twisted by a character of conductor 5 and order 4. Numerically, we see that each form from Γ^5 appear on $\Gamma_0(25)$ with multiplicity two, meaning that there should exist a map different from V_5 , taking Maass forms from Γ^5 to $\Gamma_0(25)$. Under the *assumption* that this is true, and additionally, *assuming* that all involved spaces are orthogonal, it is possible to use classical dimension formulas to show that the space of genuinely new holomorphic modular forms of even weight on $\Gamma_0(25)$ is empty.

Already in the case of $p = 7$ the situation becomes much more involved. By Cummins and Pauli [18] we know there are three conjugacy classes of groups between $\mathrm{PSL}_2(\mathbb{Z})$ and $\Gamma_0(7)$. As representatives we can choose Γ^7 (the cycloidal group), $7C^0$ (of index 14 and with two cusps) and $7F^0$ (of index 28 and with four cusps). A numerical investigation of the spectrum of $\Gamma_0(49)$ leaves a few eigenvalues which do not seem to be related to either of the groups $\mathrm{PSL}_2(\mathbb{Z})$, $\Gamma_0(7)$, Γ^7 , $7C^0$ or $7F^0$. We calculated 23 eigenvalues (counted with multiplicity) and out of these there were two eigenvalues which we could not explain using the groups mentioned above together with Dirichlet characters. Both of these eigenvalues seem to appear with multiplicity at least two, and we would therefore not expect them to be genuinely new. However, the only type of lifts which we have not investigated are those which are associated to spaces of Maass forms on the groups Γ^7 , $7C^0$ and $7F^0$ together with non-trivial characters. We have not determined whether there indeed exists non-trivial characters on these groups or not. The numerical evidence is therefore considered to be

incomplete and we do not make any conjecture in this case. Note that the two unexplained eigenvalues were also verified by D. Farmer and S. Lemurell (in connection with [19]).

We leave the matter of a more comprehensive numerical and theoretical study of genuinely new forms on the groups $\Gamma_0(N^2)$ as an open problem.

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